

Besov regularity of solutions to the p -Poisson equation ^{*}

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August 20, 2014

Abstract

In this paper, we study the regularity of solutions to the p -Poisson equation for all $1 < p < \infty$. In particular, we are interested in smoothness estimates in the adaptivity scale $B_\tau^\sigma(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$, of Besov spaces. The regularity in this scale determines the order of approximation that can be achieved by adaptive and other nonlinear approximation methods. It turns out that, especially for solutions to p -Poisson equations with homogeneous Dirichlet boundary conditions on bounded polygonal domains, the Besov regularity is significantly higher than the Sobolev regularity which justifies the use of adaptive algorithms. This type of results is obtained by combining local Hölder with global Sobolev estimates. In particular, we prove that intersections of locally weighted Hölder spaces and Sobolev spaces can be continuously embedded into the specific scale of Besov spaces we are interested in. The proof of this embedding result is based on wavelet characterizations of Besov spaces.

Keywords: p -Poisson equation, regularity of solutions, Hölder spaces, Besov spaces, nonlinear and adaptive approximation, wavelets.

Subject Classification: 35B35, 35J92, 41A25, 41A46, 46E35, 65M99, 65T60.

1 Introduction

This paper is concerned with regularity estimates of the solutions to the p -Poisson equation

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=f \quad \text{in } \Omega, \quad (1)$$

^{*}This work has been supported by Deutsche Forschungsgemeinschaft DFG (DA 360/18-1, DA 360/19-1) and European Research Council ERC (Starting Grant HDSP-CONTR-306274).

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where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ denotes some bounded Lipschitz domain. The corresponding variational formulation is given by

$$\int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle dx = \int_{\Omega} f v dx \quad \text{for all } v \in C_0^\infty(\Omega). \quad (2)$$

Problems of this type arise in many applications, e.g., in non-Newtonian fluid theory, non-Newtonian filtering, turbulent flows of a gas in porous media, rheology, radiation of heat and many others. Moreover, the p -Laplacian has a similar model character for nonlinear problems as the ordinary Laplace equation for linear problems. We refer to [36] for an introduction. By now, many results concerning existence and uniqueness of solution are known, we refer again to [36] and the references therein. However, in many cases, the concrete shape of the solutions is unknown, so that efficient numerical schemes for the constructive approximation are needed. In practice, e.g., for problems in three and more space dimensions, this might lead to systems with hundreds of thousands or even millions of unknown. Therefore, a quite natural idea would be to use *adaptive* strategies to increase efficiency. Essentially, an adaptive algorithm is an updating strategy where additional degrees of freedom are only spent in regions where the numerical approximation is still “far away” from the exact solution. Nevertheless, although the idea of adaptivity is quite convincing, these schemes are hard to analyze and to implement, so that some theoretical foundations that justify the use of adaptive strategies are highly desirable.

The analysis in this paper is motivated by this problem, in particular in connection with adaptive wavelet algorithms. In the wavelet case, there is a natural benchmark scheme for adaptivity, and that is best n -term wavelet approximation. In best n -term approximation, one does not approximate by linear spaces but by nonlinear manifolds \mathcal{M}_n , consisting of functions of the form

$$S = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda, \quad (3)$$

where $\{\psi_\lambda \mid \lambda \in \mathcal{J}\}$ denotes a given wavelet basis and $\Lambda \subset \mathcal{J}$ with $\#\Lambda = n$. We refer to Section 2 and to the textbooks [13, 39, 50] for further information concerning the construction and the basic properties of wavelets. In the wavelet setting, a best n -term approximation can be realized by extracting the n biggest wavelet coefficients from the wavelet expansion of the (unknown) function one wants to approximate. Clearly, on the one hand, such a scheme can never be realized numerically, because this would require to compute all wavelet coefficients and to select the n biggest. On the other hand, the best we can expect for an adaptive wavelet algorithm would be that it (asymptotically) realizes the approximation order of the best n -term approximation. In this sense, the use of adaptive schemes is justified if best n -term wavelet approximation realizes a significantly higher convergence order when compared to more conventional, uniform approximation schemes. In the wavelet setting, it is known that

the convergence order of uniform schemes with respect to L_p depends on the regularity of the object one wants to approximate in the scale $W^s(L_p(\Omega))$ of L_p -Sobolev spaces, whereas the order of best n -term wavelet approximation in L_p depends on the regularity in the *adaptivity scale* $B_\tau^\sigma(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$, of Besov spaces. We refer to [7, 14, 26] for further information. Therefore, the use of adaptive (wavelet) algorithms for (1) would be justified if the Besov smoothness σ of the solution in the adaptivity scale of Besov spaces is higher than its Sobolev regularity s .

For linear second order elliptic equations, a lot of positive results in this direction already exist; see, e.g., [6, 8, 10]. In contrast, it seems that not too much is known for nonlinear equations. The only contribution we are aware of is the paper [11] which is concerned with semilinear equations. In the present paper, we show a first positive result for quasilinear elliptic equations, i.e., for the p -Poisson equation (1). Results of Savaré [41] indicate that, on general Lipschitz domains, the Sobolev smoothness of the solutions to (1) is given by $s^* = 1 + 1/p$ if $2 \leq p < \infty$, and by $s^* = 3/2$ if $1 < p < 2$. However, under certain conditions, the solutions possess higher regularity away from the boundary, in the sense that they are locally Hölder continuous; see, e.g., [18, 24, 45, 48, 49]. The local Hölder semi-norms may explode as one approaches the boundary, but this singular behaviour can be controlled by some power of the distance to the boundary as shown, e.g., in [19, 32, 34, 35]. We refer to Section 4 for a detailed exposition. (Properties like this very often hold in the context of elliptic boundary problems on nonsmooth domains, we refer, e.g., to [38] and the references therein for details). It turns out that the combination of the global Sobolev smoothness and the local Hölder regularity can be used to establish Besov smoothness for the solutions to (1). In many cases, the Besov smoothness σ is much higher than the Sobolev smoothness $s^* = 1 + 1/p$ or $s^* = 3/2$ respectively, so that the use of adaptive schemes is completely justified.

We state our findings in two steps. First of all, we prove a general embedding theorem which says that the intersection of a classical Sobolev space and a Hölder space with the properties outlined above can be embedded into Besov spaces in the adaptivity scale $1/\tau = \sigma/d + 1/p$. It turns out that for a large range of parameters, the Besov smoothness is significantly higher compared to the Sobolev smoothness. The proof of this embedding theorem is performed by exploiting the characterizations of Besov spaces by means of wavelet expansion coefficients. Then we verify that under certain natural conditions the solutions to (1) indeed satisfy the assumptions of the embedding theorem, so that its application yields the desired result.

This paper is organized as follows: In Section 2, we introduce all the function spaces that will be used in the paper, including their wavelet characterizations, if possible. Afterwards, in Section 3 and Section 4, we state and prove our main results: Our general embedding (Theorem 3.1) can be found in Section 3. Its application to the case of the solutions to (1)

which yields new, generic Besov regularity results (see Theorem 4.8 and Theorem 4.15) is performed in Subsection 4.1 and 4.2, respectively. Moreover, here we give explicit bounds on the Besov regularity of the unique solution to the p -Poisson equation with homogeneous Dirichlet boundary conditions in two dimensions; see Theorem 4.17 and Theorem 4.20. The paper is concluded with an Appendix (Section 5) which contains a couple of auxiliary lemmata and propositions which are needed in our proofs.

Notation: For families $\{a_{\mathcal{J}}\}_{\mathcal{J}}$ and $\{b_{\mathcal{J}}\}_{\mathcal{J}}$ of non-negative real numbers over a common index set we write $a_{\mathcal{J}} \lesssim b_{\mathcal{J}}$ if there exists a constant $c > 0$ (independent of the context-dependent parameters \mathcal{J}) such that

$$a_{\mathcal{J}} \leq c \cdot b_{\mathcal{J}}$$

holds uniformly in \mathcal{J} . Consequently, $a_{\mathcal{J}} \sim b_{\mathcal{J}}$ means $a_{\mathcal{J}} \lesssim b_{\mathcal{J}}$ and $b_{\mathcal{J}} \lesssim a_{\mathcal{J}}$.

2 Function spaces and wavelet decompositions

In this section we recall the definitions of several types of function spaces that will be needed in the sequel. Moreover, we collect some well-known assertions such as, e.g., the characterization of Besov spaces in terms of wavelet coefficients.

2.1 Strongly differentiable functions: (weighted) Hölder spaces

Let $\Omega \subset \mathbb{R}^d$ be some bounded domain, i.e., an open and connected set. Then, for $\ell \in \mathbb{N}_0$, $C^\ell(\Omega)$ furnished with the norm

$$\|g\|_{C^\ell(\Omega)} = \sum_{|\nu| \leq \ell} \sup_{x \in \Omega} |\partial^\nu g(x)|$$

denotes the space of all real-valued functions g on Ω such that $\partial^\nu g$ is uniformly continuous and bounded on Ω for every multi-index $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$ with $0 \leq |\nu| \leq \ell$. Therein $\partial^\nu = \partial^{|\nu|} / (\partial x_1^{\nu_1} \dots \partial x_d^{\nu_d})$ denote the ν -th order strong derivatives. If K is a compact subset of Ω (denoted by $K \subset\subset \Omega$), the spaces $C^\ell(K)$ are defined likewise. Unless otherwise stated we restrict ourselves to those $K \subset\subset \Omega$ which can be described as the closure of some open and simply connected set. Next let us recall that for $g \in C^\ell(\Omega)$ the ℓ -th order Hölder semi-norm with exponent $0 < \alpha \leq 1$ is given by

$$|g|_{C^{\ell, \alpha}(\Omega)} = \sum_{|\nu| = \ell} \sup_{\substack{x, y \in \Omega, \\ x \neq y}} \frac{|\partial^\nu g(x) - \partial^\nu g(y)|}{|x - y|^\alpha}. \quad (4)$$

Consequently, for $\ell \in \mathbb{N}_0$ and $0 < \alpha \leq 1$,

$$C^{\ell,\alpha}(\Omega) = \left\{ g \in C^\ell(\Omega) \mid \|g\|_{C^{\ell,\alpha}(\Omega)} = \|g\|_{C^\ell(\Omega)} + |g|_{C^{\ell,\alpha}(\Omega)} < \infty \right\},$$

denote the (classical) *Hölder spaces* on Ω . Again we can replace Ω by K at every occurrence to define the Hölder spaces also for compact subsets $K \subset\subset \Omega$. Standard proofs yield that all the spaces we defined so far are actually Banach spaces; see, e.g., [22, 31].

Furthermore, let us introduce the collection of all functions on Ω which are locally Hölder continuous (of order $\ell \in \mathbb{N}_0$ with exponent $0 < \alpha \leq 1$). This set will be denoted by

$$C_{\text{loc}}^{\ell,\alpha}(\Omega) = \left\{ g: \Omega \rightarrow \mathbb{R} \mid g \in C^{\ell,\alpha}(K) \text{ for all } K \subset\subset \Omega \right\},$$

where we simplified the notation by denoting the restrictions $g|_K$ of functions g from Ω to compact subsets K by g again. Since the latter collection of functions does not perfectly fit for our purposes, in the sequel the following closely related (non-standard) function spaces will be used instead. Let \mathcal{K} denote an arbitrary but non-trivial family of compact subsets $K \subset\subset \Omega$. Then for every $K \in \mathcal{K}$ the quantity

$$\delta_K = \text{dist}(K, \partial\Omega), \tag{5}$$

i.e., the distance of K to the boundary of Ω , is strictly positive. Thus, for each $\ell \in \mathbb{N}_0$, all $0 < \alpha \leq 1$, and every $\gamma > 0$, the space

$$\begin{aligned} C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega; \mathcal{K}) \\ = \left\{ g: \Omega \rightarrow \mathbb{R} \mid g \in C^{\ell,\alpha}(K) \text{ for all } K \in \mathcal{K} \text{ and } |g|_{C_{\gamma,\text{loc}}^{\ell,\alpha}} = \sup_{K \in \mathcal{K}} \delta_K^\gamma |g|_{C^{\ell,\alpha}(K)} < \infty \right\} \end{aligned}$$

is well-defined and it is easily verified that $|\cdot|_{C_{\gamma,\text{loc}}^{\ell,\alpha}}$ provides a semi-norm for this space. In our applications below $\mathcal{K}(c)$ will be the set of all closed balls $B = B_r(x_0) \subset \Omega$ (with center $x_0 \in \Omega$ and radius $r > 0$) such that the (open) ball $\mathring{B}_{cr} = \mathring{B}_{cr}(x_0)$ is still contained in Ω . Here $c > 1$ denotes a constant which we assume to be given fixed in advance. Actually, it is not hard to see that the space $C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega; \mathcal{K}(c))$ is independent of c . Consequently, we simply write $C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega) = C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega; \mathcal{K})$ for $\mathcal{K} = \mathcal{K}(c)$. Those spaces are then referred to as *locally weighted Hölder spaces*.

Remark 2.1. Obviously, for every choice of the parameters, $C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$ contains $C^{\ell,\alpha}(\Omega)$ as a linear subspace, but it also contains functions g whose local Hölder semi-norms $|g|_{C^{\ell,\alpha}(K)}$ grow to infinity as the distance δ_K of $K \subset\subset \Omega$ to the boundary tends to zero. However, this possible blow-up is controlled by the parameter γ . Moreover, in the Appendix we show that

the intersection of $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ with some Besov space is a Banach space with respect to the canonical norm; see Proposition 5.3. Finally, we want to mention that the spaces $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ are monotone in γ , meaning that $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega) \subseteq C_{\mu, \text{loc}}^{\ell, \alpha}(\Omega)$ for $\gamma \leq \mu$. This can be seen by checking that $\delta_K^\mu = (\delta_K/C)^\mu C^\mu \leq (\delta_K/C)^\gamma C^\mu = \delta_K^\gamma C^{\mu-\gamma}$ for some universal constant $C \geq 1$ (e.g., $C = \max\{1, \text{diam}(\Omega)\}$), thus $|\cdot|_{C_{\mu, \text{loc}}^{\ell, \alpha}} \leq |\cdot|_{C_{\gamma, \text{loc}}^{\ell, \alpha}}$.

For the sake of completeness, we mention here that (as usual) the set of all infinitely often (strongly) differentiable functions with compact support in Ω will be denoted by $C_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$. For its dual space we write $\mathcal{D}'(\Omega)$. Once more, these definitions apply likewise when Ω is replaced by some compact set K .

2.2 Weakly differentiable functions: Sobolev spaces

Assume $\Omega \subseteq \mathbb{R}^d$ to be either \mathbb{R}^d itself, or some bounded domain. Given $0 < p \leq \infty$ the *Lebesgue spaces* $L_p(\Omega)$ consist of all (equivalence classes of real-valued) measurable functions g on Ω for which the (quasi-)norm

$$\|g\|_{L_p(\Omega)} = \begin{cases} \left(\int_{\Omega} |g(x)|^p \, dx \right)^{1/p} & \text{if } p < \infty, \\ \text{ess-sup}_{x \in \Omega} |g(x)| & \text{if } p = \infty \end{cases}$$

is finite.

Moreover, for $1 \leq p < \infty$ and $\ell \in \mathbb{N}_0$, let

$$W^\ell(L_p(\Omega)) = \left\{ g \in L_p(\Omega) \mid \|g\|_{W^\ell(L_p(\Omega))} = \sum_{|\nu| \leq \ell} \|D^\nu g\|_{L_p(\Omega)} < \infty \right\}$$

denote the classical *Sobolev spaces* on Ω , where D^ν are the weak partial derivatives of order $\nu \in \mathbb{N}_0^d$. For fractional smoothness parameters $s = \ell + \beta > 0$ (with $\ell \in \mathbb{N}_0$ and $0 < \beta < 1$) we extend the definition in the usual way by setting

$$W^s(L_p(\Omega)) = \left\{ g \in W^\ell(L_p(\Omega)) \mid \|g\|_{W^s(L_p(\Omega))} < \infty \right\},$$

where here the norm is given by $\|g\|_{W^s(L_p(\Omega))} = \|g\|_{W^\ell(L_p(\Omega))} + |g|_{W^s(L_p(\Omega))}$ and

$$|g|_{W^s(L_p(\Omega))} = \left(\sum_{|\nu|=\ell} \int_{\Omega} \int_{\Omega} \frac{|D^\nu g(x) - D^\nu g(y)|^p}{|x - y|^{d+\beta p}} \, dx \, dy \right)^{1/p}$$

denotes the common Sobolev semi-norm on Ω .

Furthermore, for $s > 0$ and $1 < p < \infty$, let us denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^s(L_p(\Omega))$ by $W_0^s(L_p(\Omega))$. Then we define $W^{-s}(L_{p'}(\Omega))$ to be the dual space of $W_0^s(L_p(\Omega))$, where p' is determined by the relation $1/p + 1/p' = 1$.

For a detailed discussion of the scale of Banach spaces $W^s(L_p(\Omega))$, $s \in \mathbb{R}$, we refer to standard textbooks such as [1, 46] and the references given therein.

2.3 Generalized smoothness: Besov spaces

A more advanced way to measure the smoothness of functions is provided by the framework of Besov spaces which essentially generalizes the concept of Sobolev spaces introduced above. Besov spaces can be defined in various ways which (for a large range of the parameters involved) lead to equivalent descriptions; cf. [3, 9, 46, 47]. For our purposes the following approach based on iterated differences seems to be the most reasonable one, since it provides an entirely *intrinsic* definition when dealing with Lipschitz domains (i.e., domains which possess a Lipschitz boundary; cf. [47, Def. 1.103]). We refer, e.g., to [4, 14, 15, 16, 17].

In the following let $\Omega \subseteq \mathbb{R}^d$ be either \mathbb{R}^d itself, or some bounded Lipschitz domain. Moreover, let $r \in \mathbb{N}$ and $h \in \mathbb{R}^d$. Then $\Omega_{r,h}$ denotes the set of all $x \in \Omega$ such that the line segment $[x, x + rh]$ belongs to Ω . Moreover, for functions g on Ω the *iterated difference* of order r with step size h is recursively given by

$$\Delta_h^1(g, x) = g(x + h) - g(x) \quad \text{and} \quad \Delta_h^r(g, x) = \Delta_h^1(\Delta_h^{r-1}(g, \cdot), x), \quad r \geq 2,$$

for every $x \in \Omega_{r,h}$. It is easily verified that

$$\Delta_h^r(g, x) = \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} g(x + kh) \quad \text{for all } r \in \mathbb{N}, h \in \mathbb{R}^d, x \in \Omega_{r,h}.$$

Those differences can be used to quantify smoothness: For $0 < p \leq \infty$ and every $g \in L_p(\Omega)$ let

$$\omega_r(g, t, \Omega)_p = \sup_{h \in \mathbb{R}^d, |h| \leq t} \|\Delta_h^r(g, \cdot)\|_{L_p(\Omega_{r,h})}, \quad t > 0,$$

denote the *modulus of smoothness* of order r . It is well-known that $\omega_r(g, t, \Omega)_p \rightarrow 0$ monotonically as t tends to zero and the faster this convergence the smoother is g .

Now let $s = \ell + \beta > 0$ with $\ell \in \mathbb{N}_0$ and $0 \leq \beta < 1$. Then, for $0 < p, q \leq \infty$, the *Besov space* $B_q^s(L_p(\Omega))$ is defined as the collection of all $g \in L_p(\Omega)$ for which the semi-norm

$$|g|_{B_q^s(L_p(\Omega))} = \begin{cases} \left(\int_0^\infty \left[t^{-s} \omega_r(g, t, \Omega)_p \right]^q \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} t^{-s} \omega_r(g, t, \Omega)_p & \text{if } q = \infty, \end{cases} \quad (6)$$

with $r \geq \ell + 1$ is finite. Endowed with the canonical (quasi-)norm

$$\|g|B_q^s(L_p(\Omega))\| = \|g|L_p(\Omega)\| + |g|_{B_q^s(L_p(\Omega))}$$

these spaces turn out to be quasi-Banach spaces (and Banach spaces if $\min\{p, q\} \geq 1$). Roughly speaking, with $\|g|B_q^s(L_p(\Omega))\|$ we can control all (weak) partial derivatives $D^\nu g$ up to the order s , measured in $L_p(\Omega)$. Since the influence of the additional *fine index* q is neglectable for many applications, we will mainly focus on the *smoothness parameter* s , as well as on the *integrability index* p , and simply set $q = p$ in what follows.

Remark 2.2. Some comments are in order:

- (i) We note that different choices of $r \geq \lfloor s \rfloor + 1$ in (6) lead to equivalent (quasi-)norms. The same is true when we restrict the range for t in (6) to the interval $(0, 1)$.
- (ii) The scale of Besov spaces as defined above is well-studied. In particular, sharp assertions on embeddings, interpolation and duality properties, characterizations in terms of various building blocks (e.g., atoms, local means, quarks, or wavelets) and best n -term approximation results are known; see, e.g., [9, 14, 17, 27]. Many of them can also be shown using the Fourier analytic definition of $B_q^s(L_p(\Omega))$ as spaces of (restrictions of) tempered distributions [25, 46, 47]. It is known [20, 42, 47] that both definitions coincide in the sense of equivalent (quasi-)norms if

$$s > \sigma_p = d \cdot \max\left\{\frac{1}{p} - 1, 0\right\}. \quad (7)$$

- (iii) The demarcation line for embeddings of Besov spaces into $L_p(\Omega)$, $1 < p < \infty$, is given by

$$\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}. \quad (8)$$

Every Besov space with smoothness and integrability indices corresponding to a point above that line is continuously embedded into $L_p(\Omega)$ (regardless of the fine index q). The points below this line never embed into $L_p(\Omega)$. For spaces $B_q^\sigma(L_\tau(\Omega))$ with (σ, τ) that satisfy (8) some care is needed. However, if $q = \tau$, then the embedding still holds. Observe that (8) exactly coincides with the adaptivity scale of Besov spaces we are interested in.

- (iv) Besov spaces are closely related to Sobolev spaces. Indeed, it has been shown that for bounded Lipschitz domains Ω , $1 \leq p < \infty$, and $0 < s \notin \mathbb{N}$ the space $B_p^s(L_p(\Omega))$

coincides with $W^s(L_p(\Omega))$ in the sense of equivalent norms; see, e.g., [17, Theorem 6.7]. Using the fact that $X^s(L_p(\Omega)) \hookrightarrow X^{s-\varepsilon}(L_p(\Omega))$ for $X \in \{B_p, W\}$ and arbitrary small $\varepsilon > 0$ we thus have

$$W^{s+\varepsilon}(L_p(\Omega)) \hookrightarrow B_p^s(L_p(\Omega)) \hookrightarrow W^{s-\varepsilon}(L_p(\Omega))$$

for all $1 \leq p < \infty$ and every $s > \varepsilon > 0$.

(v) For every bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ there exists a linear extension operator

$$\mathcal{E}_\Omega: B_q^s(L_p(\Omega)) \rightarrow B_q^s(L_p(\mathbb{R}^d))$$

which is simultaneously bounded for all parameters that satisfy (7); cf. [40]. Moreover, \mathcal{E}_Ω is local in the sense that $\text{supp}(\mathcal{E}_\Omega u)$ is contained in some bounded neighborhood of Ω ; see [9].

2.4 Wavelet characterization of Besov spaces

Under suitable conditions on the parameters involved it is possible to characterize Besov spaces by means of wavelet decompositions [13, 28, 39, 47]. These characterizations are one of the most important ingredients of wavelet analysis. In particular, they provide the basis for several numerical applications such as preconditioning and the design of adaptive algorithms. We refer to [4, 5, 7] for details. Moreover, the resulting (quasi-)norm equivalences provide a powerful tool which allows to prove continuous embeddings such as the one stated in Theorem 3.1 in Section 3 below.

To start with, we recall some basic assertions related to expansions w.r.t. Daubechies wavelets. We essentially follow the lines of [8]: Let $\{D_m \mid m \in \mathbb{N}\}$ denote the univariate family of compactly supported Daubechies wavelets [12, 13]. We remind the reader that D_m has m vanishing moments and the smoothness of these functions increases without bound as m tends to infinity. So, let us fix an arbitrary value of m and let $\psi^0 = \phi_m$ denote the univariate scaling function which generates the wavelet $\psi^1 = D_m$. Furthermore, by E we denote the non-zero vertices of the unit cube $[0, 1]^d$. Then, in dimension d , the set

$$\Psi = \Psi(d) = \left\{ \psi^e = \bigotimes_{n=1}^d \psi^{e_n} \mid e = (e_1, \dots, e_d) \in E \right\}$$

of $2^d - 1$ (tensor product) functions generates (by shifts and dilates) an orthonormal wavelet basis for $L_2(\mathbb{R}^d)$ as follows: If

$$\mathcal{I} = \mathcal{I}(\mathbb{R}^d) = \left\{ I_{j,k} = 2^{-j}k + 2^{-j}[0, 1]^d \mid k \in \mathbb{Z}^d, j \in \mathbb{Z} \right\}$$

denotes the set of all dyadic intervals in \mathbb{R}^d , then the basis consists of all functions of the form

$$\eta_I = \eta_{j,k} = 2^{jd/2} \eta(2^j \cdot -k) \quad \text{with} \quad I = I_{j,k} \in \mathcal{I}, \quad k \in \mathbb{Z}^d, \quad j \in \mathbb{Z}, \quad \text{and} \quad \eta \in \Psi. \quad (9)$$

In view of our application below, we remark that there exists some open cube $Q \subset \mathbb{R}^d$, centered at the origin with sides parallel to the coordinate axes, such that $\text{supp}(\eta) \subset Q$ for all $\eta \in \Psi$. Accordingly, all basis functions (9) satisfy $\text{supp}(\eta_I) \subset Q(I) = 2^{-j}k + 2^{-j}Q$, where

$$|Q(I)| \sim |I| = 2^{-jd} \quad \text{and} \quad Q(I) \subset B(I) = B_{2^{-(j+1)\text{diam}(Q)}}(2^{-j}k), \quad I = I_{j,k} \in \mathcal{I}. \quad (10)$$

For every $1 < q < \infty$ the system defined in (9) also forms an unconditional basis for $L_q(\mathbb{R}^d)$. Hence, for those q each $g \in L_q(\mathbb{R}^d)$ possesses a wavelet expansion

$$g = \sum_{I \in \mathcal{I}} \sum_{\eta \in \Psi} \langle g, \eta_I \rangle \eta_I \quad (11)$$

which converges in $L_q(\mathbb{R}^d)$.

For our purposes it is convenient to slightly modify this decomposition. Therefore let S_0 be the closure of all finite linear combinations of integer shifts of $\bigotimes_{n=1}^d \phi_m$ in $L_2(\mathbb{R}^d)$ and let P_0 denote the orthogonal projector which maps $L_2(\mathbb{R}^d)$ onto S_0 . Then, for every $1 < q < \infty$, the operator P_0 can be extended to a projector on $L_q(\mathbb{R}^d)$ and in (11) we can restrict ourselves to those η_I for which

$$I \in \mathcal{I}^+ = \mathcal{I}^+(\mathbb{R}^d) = \{I \in \mathcal{I}(\mathbb{R}^d) \mid |I| \leq 1\},$$

i.e., to wavelets corresponding to levels $j \in \mathbb{N}_0$. Moreover, we shall renormalize our wavelets and set

$$\eta_{I,p} = |I|^{1/2-1/p} \eta_I \quad \text{for all} \quad I \in \mathcal{I}^+, \quad \eta \in \Psi, \quad \text{and} \quad 0 < p < \infty,$$

such that $\|\eta_{I,p}\|_{L_p(\mathbb{R}^d)} = \|\eta\|_{L_p(\mathbb{R}^d)}$ does not depend on I . Incorporating these conventions, from (11) we conclude that every $g \in L_q(\mathbb{R}^d)$, $1 < q < \infty$, can be expanded as

$$\begin{aligned} g &= P_0(g) + \sum_{I \in \mathcal{I}^+} \sum_{\eta \in \Psi} \langle g, \eta_I \rangle \eta_I \\ &= P_0(g) + \sum_{I \in \mathcal{I}^+} \sum_{\eta \in \Psi} \langle g, \eta_{I,p'} \rangle \eta_{I,p}, \end{aligned} \quad (12)$$

where p' satisfies $1/p' = 1 - 1/p$.

Lemma 2.3. *Let $d \in \mathbb{N}$, $0 < p < \infty$, and $\sigma_p < s < r \in \mathbb{N}$. Moreover, choose $m \in \mathbb{N}$ such that $\phi_m, D_m \in C^r(\mathbb{R})$. Then a function g belongs to the Besov space $B_p^s(L_p(\mathbb{R}^d))$ if and only if (12) holds with*

$$\left\| P_0(g) \right\|_{L_p(\mathbb{R}^d)} + \left(\sum_{I \in \mathcal{I}^+} \sum_{\eta \in \Psi} |I|^{-sp/d} |\langle g, \eta_{I,p'} \rangle|^p \right)^{1/p} < \infty. \quad (13)$$

Furthermore, (13) provides an equivalent (quasi-)norm for $B_p^s(L_p(\mathbb{R}^d))$.

The proof of this assertion is quite standard. For the case of Banach spaces ($p \geq 1$) it can be found, e.g., in [39]. For the quasi-Banach case $0 < p < 1$ we refer to [33]. Similar assertions can also be found in [47].

Remark 2.4. We stress the point that due to $s > \sigma_p$ every $g \in B_p^s(L_p(\mathbb{R}^d))$ belongs to some $L_q(\mathbb{R}^d)$, $1 < q < \infty$, such that (12) is well-defined; see Remark 2.2(iii). Moreover, we can use the extension operator \mathcal{E}_Ω described in Remark 2.2(v) to obtain similar norm equivalences for functions in $B_p^s(L_p(\Omega))$, where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain.

As mentioned already in the introduction, we are particularly interested in Besov spaces $B_\tau^\sigma(L_\tau(\Omega))$ within the adaptivity scale of $L_p(\Omega)$, $1 < p < \infty$, i.e., spaces with parameters that satisfy (8). Therefore, we specialize Lemma 2.3 for the corresponding spaces on \mathbb{R}^d :

Proposition 2.5. *Let $d \in \mathbb{N}$, $1 < p < \infty$, as well as $0 < \sigma < r \in \mathbb{N}$, and $\tau = (\sigma/d + 1/p)^{-1}$. Moreover, choose $m \in \mathbb{N}$ such that $\phi_m, D_m \in C^r(\mathbb{R})$. Then a function g belongs to the Besov space $B_\tau^\sigma(L_\tau(\mathbb{R}^d))$ if and only if*

$$g = P_0(g) + \sum_{I \in \mathcal{I}^+} \sum_{\eta \in \Psi} \langle g, \eta_{I,p'} \rangle \eta_{I,p}$$

with

$$\left\| P_0(g) \right\|_{L_\tau(\mathbb{R}^d)} + \left(\sum_{I \in \mathcal{I}^+} \sum_{\eta \in \Psi} |\langle g, \eta_{I,p'} \rangle|^\tau \right)^{1/\tau} < \infty \quad (14)$$

and (14) provides an equivalent (quasi-)norm for $B_\tau^\sigma(L_\tau(\mathbb{R}^d))$.

Proof. Observe that $\eta_{I,\tau'} = |I|^{1/p'-1/\tau'} \eta_{I,p'}$ implies $|I|^{-\sigma\tau/d} |\langle g, \eta_{I,\tau'} \rangle|^\tau = |\langle g, \eta_{I,p'} \rangle|^\tau$. Then the proof easily follows from Lemma 2.3. \blacksquare

3 A general embedding

In this section we prove that, under some growth conditions on the local Hölder semi-norm, the intersection $B_p^s(L_p(\Omega)) \cap C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$ is continuously embedded into certain Besov spaces $B_\tau^\sigma(L_\tau(\Omega))$.

Theorem 3.1. *For $d \in \mathbb{N}$ with $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ denote some bounded Lipschitz domain. Moreover, let $s > 0$ and $1 < p < \infty$, as well as $\ell \in \mathbb{N}_0$, $0 < \alpha \leq 1$, and $0 < \gamma < \ell + \alpha + 1/p$. If we define*

$$\sigma^* = \begin{cases} \ell + \alpha & \text{if } 0 < \gamma < \frac{\ell + \alpha}{d} + \frac{1}{p}, \\ \frac{d}{d-1} \left(\ell + \alpha + \frac{1}{p} - \gamma \right) & \text{if } \frac{\ell + \alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}, \end{cases} \quad (15)$$

then for all

$$0 < \sigma < \min \left\{ \sigma^*, \frac{d}{d-1} s \right\} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p} \quad (16)$$

we have the continuous embedding

$$B_p^s(L_p(\Omega)) \cap C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega) \hookrightarrow B_\tau^\sigma(L_\tau(\Omega)),$$

i.e., for all $u \in B_p^s(L_p(\Omega)) \cap C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$ it holds

$$\|u\|_{B_\tau^\sigma(L_\tau(\Omega))} \lesssim \max \left\{ \|u\|_{B_p^s(L_p(\Omega))}, |u|_{C_{\gamma,\text{loc}}^{\ell,\alpha}} \right\}. \quad (17)$$

Let us briefly comment on Theorem 3.1 before we give its proof: From the theory of function spaces it is well-known that (standard) embeddings between Besov spaces, e.g.,

$$B_p^s(L_p(\Omega)) \hookrightarrow B_\tau^\sigma(L_\tau(\Omega)),$$

are valid *only if* the regularity of the target space is at most as large as the smoothness of the space we start from, i.e., only if $\sigma \leq s$. Theorem 3.1 now states that, under suitable assumptions on the parameters involved, exploiting the additional information on locally weighted Hölder regularity (encoded by the membership of u in $C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$) enables us to prove that functions from $B_p^s(L_p(\Omega))$ indeed possess a higher-order Besov regularity $\sigma > s$ measured in the adaptivity scale corresponding to $L_p(\Omega)$. Since $B_p^s(L_p(\Omega))$ almost equals the Sobolev space $W^s(L_p(\Omega))$ (cf. Remark 2.2(iv)) this shows that approximating $u \in W^s(L_p(\Omega)) \cap C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$ in an adaptive way is justified whenever σ^* defined by (15) is larger than s . At this

point we remark that σ^* is a continuous piecewise linear function of $\gamma \in (0, \ell + \alpha + 1/p)$ which decreases to zero when γ approaches its upper bound. Hence, in any case $0 < \sigma^* \leq \ell + \alpha$. Thus, for a fixed value of s , the maximal regularity $d/(d-1) \cdot s$ is achieved if $\ell + \alpha$ is sufficiently large and γ is small enough.

The proof of Theorem 3.1 given below is inspired by ideas first given in [8]. Due to extension arguments in conjunction with the wavelet characterization of Besov spaces on \mathbb{R}^d (see Remark 2.4) it suffices to find suitable estimates for the wavelet coefficients $\langle u, \eta_{I,p'} \rangle$, $I \in \mathcal{I}^+$, $\eta \in \Psi$, which then imply (17). The contribution of (the relatively small number of) wavelets supported in the vicinity of the boundary of Ω (*boundary wavelets*) can be bounded in terms of the norm of u in $B_p^s(L_p(\Omega))$. Here the restriction $\sigma < s \cdot d/(d-1)$ comes in. The coefficients corresponding to the remaining *interior wavelets* can be upper bounded by the semi-norm of u in $C_{\gamma, \text{loc}}^{\ell, \alpha}$ using a Whitney-type argument which then gives rise to the restriction $\sigma < \sigma^*$. The detailed proof reads as follows:

Proof (of Theorem 3.1). Step 1. Let $u \in B_p^s(L_p(\Omega)) \cap C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$. Since for $1 < p < \infty$ it is $\sigma_p = 0$ and $s > 0$, every such u can be extended to some $\mathcal{E}_\Omega u \in B_p^s(L_p(\mathbb{R}^d))$; see Remark 2.2(v). In particular, $\mathcal{E}_\Omega u \in L_p(\mathbb{R}^d)$ such that it can be written as

$$\mathcal{E}_\Omega u = P_0(\mathcal{E}_\Omega u) + \sum_{(I, \eta) \in \mathcal{I}^+ \times \Psi} \langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle \eta_{I,p'}.$$

Here the η_I form a system of Daubechies wavelets (9), where $m \in \mathbb{N}$ is chosen such that $m > \ell$ and $\phi_m, D_m \in C^r(\mathbb{R})$ for some $r \in \mathbb{N}$ with $r > \max\{\sigma, s\}$; see Subsection 2.4 for details. We restrict the latter expansion and consider only those wavelets for which (I, η) belongs to

$$\Lambda = \bigcup_{j \in \mathbb{N}_0} \Lambda_j, \quad \text{where we set} \quad \Lambda_j = \left\{ (I, \eta) \in \mathcal{I}^+ \times \Psi \mid B_c(I) \cap \Omega \neq \emptyset \text{ and } |I| = 2^{-jd} \right\}.$$

Therein $B_c(I)$ denotes the ball $B(I)$ (see (10)) concentrically expanded by the factor $c > 1$ which we used to define the class $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$; cf. Subsection 2.1. Note that thus $\text{supp}(\eta_I) \subset B_c(I)$ for all I and η . Next we split up the index sets Λ_j once more and write

$$\Lambda_j = \bigcup_{n \in \mathbb{N}_0} \Lambda_{j,n} \quad \text{with} \quad \Lambda_{j,n} = \left\{ (I_{j,k}, \eta) \in \Lambda_j \mid n 2^{-j} \leq \text{dist}(2^{-j}k, \partial\Omega) < (n+1) 2^{-j} \right\},$$

for every dyadic level $j \in \mathbb{N}_0$. Note that, due to the boundedness of Ω , there exists an absolute constant C_1 such that $\Lambda_{j,n} = \emptyset$ for all $j \in \mathbb{N}_0$ and $n > C_1 2^j$. For example, we may take $C_1 = \max\{\text{diam}(\Omega), c \text{diam}(Q)\}$. Moreover, our assumption that Ω is a bounded Lipschitz domain ensures that all remaining index sets satisfy at least $|\Lambda_{j,n}| \lesssim 2^{-j(d+1)}$.

Finally, we note that all balls $B_c(I)$ corresponding to $(I, \eta) \in \Lambda_{j,n}$ with $j \in \mathbb{N}_0$ and n strictly larger than $C_0 = \lceil c \operatorname{diam}(Q)/2 \rceil$ are completely contained in Ω . These considerations justify the disjoint splitting $\Lambda = \left(\bigcup_{j \in \mathbb{N}_0} \Lambda_j^{\text{bnd}} \right) \cup \left(\bigcup_{j \in \mathbb{N}_0} \Lambda_j^{\text{int}} \right)$, where

$$\Lambda_j^{\text{bnd}} = \bigcup_{n=0}^{C_0} \Lambda_{j,n} \quad \text{and} \quad \Lambda_j^{\text{int}} = \bigcup_{n=C_0+1}^{C_1 2^j} \Lambda_{j,n}$$

correspond to the sets of boundary and interior wavelets at level $j \in \mathbb{N}_0$, respectively. Observe that then $\tilde{u} = u_0 + u_1 + u_2$, defined by

$$u_0 = P_0(\mathcal{E}_\Omega u), \quad u_1 = \sum_{j \in \mathbb{N}_0} \sum_{(I, \eta) \in \Lambda_j^{\text{bnd}}} \langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle \eta_{I,p}, \quad \text{and} \quad u_2 = \sum_{j \in \mathbb{N}_0} \sum_{(I, \eta) \in \Lambda_j^{\text{int}}} \langle u, \eta_{I,p'} \rangle \eta_{I,p},$$

is an extension of u as well, i.e., it satisfies $\tilde{u}|_\Omega = u$. In Step 2–4 below we will show that for the adaptivity scale $\tau = (\sigma/d + 1/p)^{-1}$ it holds

$$\|u_0\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} \lesssim \|P_0(\mathcal{E}_\Omega u)\|_{L_p(\mathbb{R}^d)} \quad \text{if } 0 < \sigma, \quad (18)$$

$$\|u_1\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} \lesssim \left[\sum_{j \in \mathbb{N}_0} \sum_{(I, \eta) \in \Lambda_j^{\text{bnd}}} |I|^{-s p/d} |\langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle|^p \right]^{1/p} \quad \text{if } 0 < \sigma < \frac{d}{d-1} s, \text{ and} \quad (19)$$

$$\|u_2\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} \lesssim |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}} \quad \text{if } 0 < \sigma < \sigma^*. \quad (20)$$

Suppose we already know that those relations hold for all σ and τ that satisfy (16). Then we can extend the index set in (19) from $\bigcup_{j \in \mathbb{N}_0} \Lambda_j^{\text{bnd}}$ to $\mathcal{I}^+ \times \Psi$ and the wavelet characterization of $\mathcal{E}_\Omega u \in B_p^s(L_p(\mathbb{R}^d))$ (cf. Lemma 2.3) together with the continuity of \mathcal{E}_Ω implies

$$\|u_0 + u_1\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} \lesssim \|\mathcal{E}_\Omega u\|_{B_p^s(L_p(\mathbb{R}^d))} \sim \|u\|_{B_p^s(L_p(\Omega))} \quad (21)$$

which is finite due to our assumptions. Therefore, the special choice $g = \tilde{u} = (u_0 + u_1) + u_2$, in conjunction with (20) and (21), yields the desired estimate

$$\begin{aligned} \|u\|_{B_\tau^\sigma(L_\tau(\Omega))} &\sim \inf \left\{ \|g\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} \mid g \in B_\tau^\sigma(L_\tau(\mathbb{R}^d)) \text{ with } g|_\Omega = u \right\} \\ &\lesssim \|u_0 + u_1\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} + \|u_2\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} \\ &\lesssim \max \left\{ \|u\|_{B_p^s(L_p(\Omega))}, |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}} \right\}. \end{aligned}$$

This proves Theorem 3.1 since $u \in B_p^s(L_p(\Omega))$ with $s > 0 = \sigma_p$ particularly implies that $u \in L_p(\Omega) \hookrightarrow L_\tau(\Omega)$, due to $\tau < p$ and the boundedness of Ω . Hence, $u \in B_\tau^\sigma(L_\tau(\Omega))$.

Step 2 (Estimate for u_0). To show the bound on the projection onto the coarse levels let $\tau = (\sigma/d + 1/p)^{-1}$ and $\sigma > 0$. We note that $u_0 \perp \eta_{I,p'}$ for all $I \in \mathcal{I}^+$ and $\eta \in \Psi$, i.e., $u_0 = P_0(u_0)$. Moreover, by definition, this equals $P_0(\mathcal{E}_\Omega u)$ which has compact support in \mathbb{R}^d since \mathcal{E}_Ω is local; see Remark 2.2(v). Proposition 2.5, i.e., the wavelet characterization of $B_\tau^\sigma(L_\tau(\mathbb{R}^d))$, therefore gives

$$\|u_0\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))} \sim \|P_0(\mathcal{E}_\Omega u)\|_{L_\tau(\mathbb{R}^d)} \lesssim \|P_0(\mathcal{E}_\Omega u)\|_{L_p(\mathbb{R}^d)},$$

due to $\tau < p$. That is, we have shown (18).

Step 3 (Estimate for u_1). Here we establish the bound on the contribution of all wavelets near $\partial\Omega$. To this end, assume again that $\tau = (\sigma/d + 1/p)^{-1}$ with $\sigma > 0$. We fix $j \in \mathbb{N}_0$ for a moment and apply Hölder's inequality (with $q = p/\tau > 1$) to estimate

$$\begin{aligned} \sum_{(I,\eta) \in \Lambda_j^{\text{bnd}}} |\langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle|^\tau &\leq |\Lambda_j^{\text{bnd}}|^{1-\tau/p} \left(\sum_{(I,\eta) \in \Lambda_j^{\text{bnd}}} |\langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle|^p \right)^{\tau/p} \\ &\lesssim 2^{j(d-1)(1-\tau/p)} 2^{-js\tau} \left(\sum_{(I,\eta) \in \Lambda_j^{\text{bnd}}} |I|^{-sp/d} |\langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle|^p \right)^{\tau/p}. \end{aligned}$$

Taking the sum over all levels j and using Hölder's inequality once more (with the same q), we find

$$\begin{aligned} \sum_{j \in \mathbb{N}_0} \sum_{(I,\eta) \in \Lambda_j^{\text{bnd}}} |\langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle|^\tau &\tag{22} \\ &\lesssim \left(\sum_{j \in \mathbb{N}_0} \left[2^{(d-1)-s\tau/(1-\tau/p)} \right]^j \right)^{1-\tau/p} \left(\sum_{j \in \mathbb{N}_0} \sum_{(I,\eta) \in \Lambda_j^{\text{bnd}}} |I|^{-sp/d} |\langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle|^p \right)^{\tau/p} \\ &\lesssim \left(\sum_{j \in \mathbb{N}_0} \sum_{(I,\eta) \in \Lambda_j^{\text{bnd}}} |I|^{-sp/d} |\langle \mathcal{E}_\Omega u, \eta_{I,p'} \rangle|^p \right)^{\tau/p}, \end{aligned}$$

provided that we additionally assume

$$\sigma < \frac{d}{d-1} s,$$

since this condition is equivalent to $1/\tau < s/(d-1) + 1/p$ which in turn holds if and only if $(d-1) - s\tau/(1-\tau/p) < 0$. Finally, the structure of u_1 together with Proposition 2.5 shows that the quantity (22) is equivalent to $\|u_1\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))}^\tau$ such that (19) follows.

Step 4 (Estimate for u_2). We are left with the proof of (20), i.e., the bound for the interior wavelets indexed by $(I, \eta) \in \bigcup_{j \in \mathbb{N}_0} \Lambda_j^{\text{int}}$. Recall that $\eta_{I,p'}$ is orthogonal to every polynomial \mathcal{P} of total degree strictly less than m . Therefore, for all (I, η) under consideration,

$$|\langle u, \eta_{I,p'} \rangle| = |\langle u - \mathcal{P}, \eta_{I,p'} \rangle| \leq \|u - \mathcal{P}\|_{L_p(Q(I))} \cdot \|\eta_{I,p'}\|_{L_{p'}(Q(I))} \lesssim \|u - \mathcal{P}\|_{L_p(Q(I))}.$$

Consequently, a Whitney-type argument (i.e., the application of Proposition 5.1 stated in the Appendix with $t = \ell + \alpha$ and $q = \infty$) shows that

$$|\langle u, \eta_{I,p'} \rangle| \lesssim \inf_{\mathcal{P} \in \Pi_\ell} \|u - \mathcal{P}\|_{L_p(Q(I))} \lesssim |Q(I)|^{(\ell+\alpha)/d+1/p} |u|_{B_\infty^{\ell+\alpha}(L_\infty(Q(I)))},$$

since we assumed $m > \ell$. Next we use (10) and estimate the Besov semi-norm by the Hölder semi-norm (see Proposition 5.2) to obtain

$$\begin{aligned} |\langle u, \eta_{I,p'} \rangle| &\lesssim 2^{-j(\ell+\alpha+d/p)} |u|_{C^{\ell,\alpha}(Q(I))} \\ &\lesssim 2^{-j(\ell+\alpha+d/p)} \delta_{B(I)}^{-\gamma} |u|_{C_{\gamma,\text{loc}}^{\ell,\alpha}} \quad \text{for all } (I, \eta) \in \bigcup_{j \in \mathbb{N}_0} \Lambda_j^{\text{int}} = \bigcup_{j \in \mathbb{N}_0} \bigcup_{n=C_0+1}^{C_1 2^j} \Lambda_{j,n}, \end{aligned} \quad (23)$$

because the open cubes $Q(I)$ are contained in the closed balls $B(I)$ by definition. For fixed $j \in \mathbb{N}_0$, $n \in \{C_0 + 1, C_0 + 2, \dots, C_1 2^j\}$, and $(I, \eta) \in \Lambda_{j,n}$, we have

$$\delta_{B(I)} \geq \delta_{B_c(I)} \geq \text{dist}(2^{-j}k, \partial\Omega) - \frac{c \text{diam}(Q)}{2} 2^{-j} \geq (n - C_0) 2^{-j}. \quad (24)$$

Now let $\tau > 0$ and recall the estimate $|\Lambda_{j,n}| \lesssim 2^{j(d-1)}$ which we found in Step 1. Combining this with (23) and (24) thus yields

$$\begin{aligned} \sum_{(I,\eta) \in \Lambda_j^{\text{int}}} |\langle u, \eta_{I,p'} \rangle|^\tau &\lesssim \sum_{n=C_0+1}^{C_1 2^j} \sum_{(I,\eta) \in \Lambda_{j,n}} 2^{-j(\ell+\alpha+d/p)\tau} (n - C_0)^{-\gamma\tau} 2^{j\gamma\tau} |u|_{C_{\gamma,\text{loc}}^{\ell,\alpha}}^\tau \\ &\lesssim |u|_{C_{\gamma,\text{loc}}^{\ell,\alpha}}^\tau 2^{-j(\ell+\alpha+d/p-\gamma)\tau+j(d-1)} \sum_{t=1}^{C_1 2^j} t^{-\gamma\tau}, \quad j \in \mathbb{N}_0. \end{aligned} \quad (25)$$

Note that, due to the assumption $\gamma > 0$, the quantity $\gamma\tau$ is always positive. Then straightforward calculations show that for all $j \in \mathbb{N}_0$

$$1 \leq \sum_{t=1}^{C_1 2^j} t^{-\gamma\tau} \lesssim \begin{cases} 2^{j(1-\gamma\tau)} & \text{if } \gamma\tau \in (0, 1), \\ 1 + j & \text{if } \gamma\tau = 1, \\ 1 & \text{if } \gamma\tau > 1, \end{cases}$$

such that we have to distinguish several cases for γ in what follows:

Substep 4.1 (Small γ). Let us consider the case $0 < \gamma < (\ell + \alpha)/d + 1/p$ first. Then obviously $d(\gamma - 1/p) < \ell + \alpha$, such that we can set

$$\tau = \left(\frac{\sigma}{d} + \frac{1}{p} \right)^{-1} \quad \text{with} \quad \max \left\{ 0, d \left(\gamma - \frac{1}{p} \right) \right\} < \sigma < \ell + \alpha. \quad (26)$$

From $d(\gamma - 1/p) < \sigma$ we particularly infer that $\gamma < \tau^{-1}$, i.e., $\gamma\tau < 1$, for this choice of τ . Therefore, from the considerations stated above we conclude that

$$\begin{aligned} \sum_{j \in \mathbb{N}_0} \sum_{(I, \eta) \in \Lambda_j^{\text{int}}} |\langle u, \eta_{I, p'} \rangle|^\tau &\lesssim |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}}^\tau \sum_{j \in \mathbb{N}_0} 2^{-j(\ell + \alpha + d/p - \gamma)\tau + j(d-1) + j(1-\gamma\tau)} \\ &= |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}}^\tau \sum_{j \in \mathbb{N}_0} \left(2^{d - (\ell + \alpha + d/p)\tau} \right)^j \\ &\lesssim |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}}^\tau, \end{aligned}$$

because the sum in the second line converges for $d - (\ell + \alpha + d/p)\tau < 0$ which is equivalent to $\sigma < \ell + \alpha = \sigma^*$. Similar to the end of Step 3, we note that the double sum on the left-hand side is equivalent to $\|u_2\|_{B_\tau^\sigma(L_\tau(\mathbb{R}^d))}^\tau$ such that (20) follows (in the case of small γ) for all σ that satisfy (26). Note that if $\gamma > 1/p$, then the maximum in (26) is strictly positive. The result (20) for $\sigma > 0$ below this value can be deduced from the assertion we just proved by means of the standard embedding along the adaptivity scale:

$$B_{\tau_2}^{\sigma_2}(L_{\tau_2}(\mathbb{R}^d)) \hookrightarrow B_{\tau_1}^{\sigma_1}(L_{\tau_1}(\mathbb{R}^d)) \quad \text{for all} \quad \sigma_2 \geq \sigma_1 > 0,$$

where $1/\tau_i = \sigma_i/d + 1/p$ for each $i \in \{1, 2\}$.

Substep 4.2 (Large γ). We turn to the case

$$\frac{\ell + \alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}.$$

As mentioned right after the statement of Theorem 3.1, for γ in this range we have that

$$\sigma^* = \frac{d}{d-1} \left(\ell + \alpha + \frac{1}{p} - \gamma \right) \leq \ell + \alpha.$$

The lower bound for γ thus implies that $\sigma^* \leq d\gamma - d/p$. Therefore, for every $0 < \sigma < \sigma^*$ the corresponding τ in the adaptivity scale satisfies

$$\frac{1}{p} < \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p} < \gamma,$$

i.e., $\gamma\tau > 1$. Hence, proceeding as in the previous substep yields

$$\begin{aligned} \sum_{j \in \mathbb{N}_0} \sum_{(I, \eta) \in \Lambda_j^{\text{int}}} |\langle u, \eta_{I, p'} \rangle|^\tau &\lesssim |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}}^\tau \sum_{j \in \mathbb{N}_0} 2^{-j(\ell + \alpha + d/p - \gamma)\tau + j(d-1)} \\ &= |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}}^\tau \sum_{j \in \mathbb{N}_0} \left(2^{d-1-\tau(\ell + \alpha + d/p - \gamma)} \right)^j \lesssim |u|_{C_{\gamma, \text{loc}}^{\ell, \alpha}}^\tau, \end{aligned}$$

where this time the sum over j converges if $d - 1 - \tau(\ell + \alpha + d/p - \gamma) < 0$ which is (for the assumed range of γ) equivalent to $\sigma < \sigma^*$. Since this implies the desired estimate (20), finally, the proof is complete. \blacksquare

Remark 3.2. The interested reader might ask what happens if $\gamma \geq \ell + \alpha + 1/p$. For $\gamma \geq \ell + \alpha + d/p$ the sum over (25) w.r.t. $j \in \mathbb{N}_0$ can never be convergent, because due to $\tau > 0$ the exponent $-j(\ell + \alpha + d/p - \gamma)\tau + j(d-1)$ would be non-negative for all j and the sum over t is bounded from below by 1. Hence, we are left with $\ell + \alpha + 1/p \leq \gamma < \ell + \alpha + d/p$. Choosing $\tau > 0$ such that $\gamma \leq 1/\tau$ then implies $\sigma \geq d(\ell + \alpha)$ for σ in the adaptivity scale. On the other hand, $\sigma < \ell + \alpha$ would be necessary for the geometric series to converge; see Substep 4.1. In contrast, if we choose $\tau > 0$ such that $\gamma > 1/\tau$, then convergence is equivalent to $\sigma < \frac{d}{d-1}(\ell + \alpha + 1/p - \gamma)$ which contradicts $\sigma > 0$ for the range of γ under consideration.

4 Besov regularity

This section is concerned with the regularity of solutions to the p -Poisson equation (1), $1 < p < \infty$, in the adaptivity scale of Besov spaces $B_\tau^\sigma(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$. In Subsection 4.1 we deal with the general case of multidimensional, bounded Lipschitz domains. The main result of this part, Theorem 4.8, describes (generic) sufficient conditions on the parameters of locally weighted Hölder spaces which ensure that the Besov regularity of all solutions u to (1) that are contained in such spaces exceeds the Sobolev smoothness of u . Subsection 4.2 then is devoted to problems on two-dimensional domains, since there many more results concerning local Hölder regularity are available in the literature. Among other things, in this subsection, we state and prove explicit Besov regularity assertions for the unique solution to the p -Poisson equation (1), with a right-hand in $L_q(\Omega)$, $q \geq p'$, which satisfies a homogeneous Dirichlet boundary condition. These statements constitute the main results of the present paper. In Theorem 4.17 we deal with general bounded Lipschitz domains $\Omega \subset \mathbb{R}^2$, whereas Theorem 4.20 contains the results for the special case of bounded polygonal domains.

Existence and uniqueness of weak solutions to all problems we are going to consider is guaranteed by the following fairly general result which is well-known in the literature. Its proof can be found, e.g., in Lions [37, Chapter 2].

Proposition 4.1 (Existence and uniqueness). *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ denote a bounded domain and let $1 < p < \infty$. Moreover, assume $f \in W^{-1}(L_{p'}(\Omega))$, as well as $g \in W^1(L_p(\Omega))$. Then the problem*

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f \quad \text{in } \Omega, \\ u - g &\in W_0^1(L_p(\Omega)), \end{aligned} \tag{27}$$

admits a unique weak solution $u \in W^1(L_p(\Omega))$.

Remark 4.2. Note that, since we like to deal with bounded Lipschitz domains Ω and $q \geq p'$, the chain of embeddings

$$L_q(\Omega) \hookrightarrow L_{p'}(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$$

together with Proposition 4.1 (applied for $g \equiv 0$) guarantees that there is at least one $u \in W^1(L_p(\Omega))$ that solves the p -Poisson equation (1) with $f \in L_q(\Omega)$.

In order to prove non-trivial Besov regularity results, we will make use of the general embedding Theorem 3.1. For that reason, we need to determine preferably small spaces $B_p^s(L_p(\Omega))$ and $C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$ which still contain the solution u to the respective problem under consideration. Clearly, smoothness results w.r.t. the Besov scale $B_p^s(L_p(\Omega))$ can be derived easily from corresponding Sobolev regularity assertions using the intimate relation of Sobolev and Besov spaces described in Remark 2.2(iv). The local Hölder regularity of solutions to the p -Poisson equation (1), as well as to more general quasi-linear elliptic problems, was studied in several papers. We refer, e.g., to Ural'ceva [49], Uhlenbeck [48], Evans [24], Lewis [34], DiBenedetto [18], Tolksdorf [45], Diening, Kaplický and Schwarzacher [19], Kuusi and Mingione [32], as well as to Teixeira [44]. The subsequent proposition can be derived as a special case from [19, Corollary 5.5] (see also [19, Remark 5.7]).

Proposition 4.3 ($C_{\text{loc}}^{1,\alpha}(\Omega)$ regularity). *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ denote any bounded domain, let $1 < p < \infty$, and $q > d$. Then there exists $\alpha \in (0, 1)$ such that all $u \in W^1(L_p(\Omega))$ which are weak solutions to (1) with $f \in L_q(\Omega)$ belong to $C_{\text{loc}}^{\ell,\alpha}(\Omega)$.*

Remark 4.4. It is well-known that, for $p > 2$, solutions to (1) do not possess continuous second derivatives in general, even if f is smooth. For instance, a weak solution to the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 1 \quad \text{on } \mathring{B}_1(0)$$

is given by

$$u(x_1, \dots, x_d) = \frac{p}{p-1} |x_1|^{p/(p-1)},$$

see [43, Proposition 5.4] and [35]. Hence, in this respect $\ell = 1$ in Proposition 4.3 is sharp at least for $p > 2$.

Here and in what follows we shall say a given problem is of *sharp regularity* α if α is a lower bound for the smoothness (measured in a certain scale) of *all* solutions to *any* problem instances (e.g., for all Lipschitz domains Ω and each $f \in L_{p'}(\Omega)$), but for every $\varepsilon > 0$ there exists a problem instance such that its corresponding solution has a regularity strictly less than $\alpha^* + \varepsilon$.

4.1 The p -Poisson equation in arbitrary dimensions

Regularity results for partial differential equations are usually stated in terms of shift theorems. Concerning the p -Poisson equation (1) with homogeneous Dirichlet boundary conditions,

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{28}$$

and the scale of Sobolev spaces $W^s(L_p(\Omega))$ one such result is due to Savaré [41, Theorems 2 and 2']:

Proposition 4.5 (Sobolev regularity on Lipschitz domains). *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Given $1 < p < \infty$ and $f \in W^{-1}(L_{p'}(\Omega))$ let $u \in W_0^1(L_p(\Omega))$ denote the unique solution to (28). Then, for $\theta \in [0, 1)$,*

$$f \in W^{t_\theta}(L_{p'}(\Omega)) \quad \text{with} \quad t_\theta = \begin{cases} -1 + \theta/2 & \text{if } 1 < p \leq 2, \\ -1 + \theta/p' & \text{if } 2 < p < \infty \end{cases} \tag{29}$$

implies that

$$u \in W^{s_\theta}(L_p(\Omega)) \quad \text{with} \quad s_\theta = \begin{cases} 1 + \theta/2 & \text{if } 1 < p \leq 2, \\ 1 + \theta/p & \text{if } 2 < p < \infty. \end{cases}$$

Remark 4.6. In [41, Remark 4.3] Savaré states that the regularity results given in Proposition 4.5 are sharp (in the sense defined above), even for the class of smooth domains.

Observe that $L_{p'}(\Omega) \hookrightarrow W^{t_\theta}(L_{p'}(\Omega))$ for all θ under consideration. Hence, provided that $f \in L_{p'}(\Omega)$, the preceding Proposition 4.5 shows that the unique solution to (28) is contained in $W^s(L_p(\Omega))$ and in $B_p^s(L_p(\Omega))$, respectively, for all $s < s^*$, where we set

$$s^* = \begin{cases} 3/2 & \text{if } 1 < p \leq 2, \\ 1 + 1/p & \text{if } 2 < p < \infty. \end{cases} \quad (30)$$

Moreover, let us mention that Savaré actually proved (for an even larger class of equations and slightly weaker assumptions on f) that we may replace $B_p^s(L_p(\Omega))$, $s < s^*$, by $B_\infty^{s^*}(L_p(\Omega))$. However, this slightly stronger assertion would not provide any gain in what follows.

Remark 4.7. In addition to Remark 4.6 we state that there are good reasons to assume that s^* given in (30) defines a sharp bound for the Sobolev regularity of solutions u to (28), even for much smoother right-hand sides f . First of all, this conjecture is supported by the well-known fact that there exist Lipschitz domains Ω such that the solution for $p = 2$ and some $f \in C^\infty(\overline{\Omega})$ does not belong to any $W^{3/2+\varepsilon}(L_2(\Omega))$, $\varepsilon > 0$; see, e.g., Jerison and Kenig [30, Theorem A]. Moreover, for $d = 2$ and $p > 2$ it can be seen easily that $s^* = 1 + 1/p$ can not be improved for general Lipschitz domains, as the following example shows: Given $\omega \in (0, 2\pi)$ let

$$\mathfrak{C}(\omega) = \{(r, \theta) \in [0, \infty) \times [0, 2\pi] \mid 0 < r < 1 \text{ and } 0 < \theta < \omega\}$$

denote an (open) circular sector of radius 1 which is centered at the origin and possesses a central angle ω . Then, by [21, Theorem 3] (see also [2]), there exist $\alpha(\omega) > 0$ which can be computed explicitly and some function t such that, under quite mild conditions on the right-hand side f , for every solution u to (28) in $\Omega = \mathfrak{C}(\omega)$ there exist a positive constant k and a function v such that

$$u(r, \theta) = k \cdot r^{\alpha(\omega)} t(\theta) + v(r, \theta), \quad (r, \theta) \in \mathfrak{C}(\omega), \quad (31)$$

where v fulfills

$$|v(r, \theta)| \lesssim r^{\alpha(\omega)+\eta} \quad \text{and} \quad |\nabla v(r, \theta)| \lesssim r^{\alpha(\omega)+\eta} \quad (32)$$

for some absolute constant $\eta > 0$. It follows from (31), (32), and the special structure of $t(\theta)$, cf. [21, Theorem 1], that $|\nabla u(r, \theta)| \sim r^{\alpha(\omega)-1}$ near the origin. Therefore $|\nabla u| \in L_\mu(\mathfrak{C}(\omega))$ can hold true only if $\mu \cdot (\alpha(\omega) - 1) > -2$. On the other hand, the behaviour of $\alpha(\omega)$ for large central angles ω , is known: It has been shown that

$$\lim_{\omega \rightarrow 2\pi} \alpha(\omega) = \frac{p-1}{p}. \quad (33)$$

Hence, by (33), for every $\mu > 2p$ there exists a two-dimensional Lipschitz domain $\Omega = \mathfrak{C}(\omega)$ and a solution u to (28) such that $|\nabla u|$ does not belong to $L_\mu(\Omega)$. Consequently, for this solution Sobolev's embedding yields that $|\nabla u|$ is not contained in $W^{1/p+\varepsilon}(L_p(\Omega))$ for any $\varepsilon > 0$ and thus

$$u \notin W^{1+1/p+\varepsilon}(L_p(\Omega)). \quad (34)$$

Finally, let us remark that for the open circular sector with $\omega = 2\pi$ the same arguments yield (34) with $\varepsilon = 0$. However, note that then $\Omega = \mathfrak{C}(2\pi)$ is not a Lipschitz domain anymore.

Unfortunately, if $d \geq 3$, then (to our best knowledge) finding the sharp local Hölder regularity α of solutions to (1), (27), or (28), respectively, still is an open problem. Moreover, in the articles mentioned before the statement of Proposition 4.3, there appear too many unspecified constants that do not seem to allow estimates for the local Hölder semi-norms which are sufficient for our purposes, i.e., to obtain a satisfactory bound for the parameter γ . In contrast, for the case $d = 2$ much more explicit results are available such that these two drawbacks can be resolved. Consequently, we present a detailed discussion of the two-dimensional case in Subsection 4.2. To conclude the current subsection, at least we want to determine the *range* of the parameters α and γ for which the Besov regularity of the solution u (in the general multidimensional setting) *would* exceed its Sobolev regularity.

Theorem 4.8. *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ denote a bounded Lipschitz domain. Moreover, for $1 < p < \infty$ and $f \in W^{-1}(L_{p'}(\Omega))$ let u be a weak solution to (1) which satisfies $u \in W^s(L_p(\Omega))$ for all $s < \bar{s} \in [\ell, \ell + 1)$ with some $\ell \in \mathbb{N}$. If, additionally, u is contained in $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ with*

$$\bar{s} - \ell < \alpha \leq 1 \quad \text{and} \quad 0 < \gamma < \ell + \alpha + \frac{1}{p} - \frac{d-1}{d} \bar{s}, \quad (35)$$

then there exists $\bar{\sigma} > \bar{s}$ such that

$$u \in B_\tau^\sigma(L_\tau(\Omega)) \quad \text{for all} \quad 0 < \sigma < \bar{\sigma} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}.$$

Before proving Theorem 4.8 we stress that, according to Proposition 4.1, we know that there indeed exists $\bar{s} \geq 1$ such that all solutions u to the p -Poisson equation (1) are contained in $W^s(L_p(\Omega))$ for all $s < \bar{s}$. Moreover, at least when dealing with homogeneous boundary conditions (i.e., solutions of (28)), it is reasonable to assume that $s \in [\ell, \ell + 1)$ and that $u \in C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ with $\ell = 1$; see Remark 4.7 and Remark 4.4, respectively. Hence, Theorem 4.8 particularly describes a wide range of sufficient conditions which ensure that the Besov regularity σ (measured in the adaptivity scale w.r.t. $L_p(\Omega)$) of solutions u to (28) on bounded Lipschitz domains is strictly larger than its maximal Sobolev regularity \bar{s} . Moreover, we note that the upper bound $\bar{\sigma}$ can be calculated (from p , the regularity parameters ℓ , α , and γ , as well as the dimension d), as the following proof shows.

Proof (of Theorem 4.8). Since we assume that $u \in W^s(L_p(\Omega))$, $s < \bar{s}$, standard embeddings (cf. Remark 2.2) imply that $u \in B_p^s(L_p(\Omega))$ for all $s \in (0, \bar{s})$. Then, for general $0 < \alpha \leq 1$ and $0 < \gamma < \ell + \alpha + 1/p$, our embedding result (Theorem 3.1) states that the additional assumption $u \in C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ yields $u \in B_\tau^\sigma(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$, for all

$$0 < \sigma < \min \left\{ \sigma^*, \frac{d}{d-1} (\bar{s} - \varepsilon) \right\} =: \bar{\sigma},$$

where $\varepsilon > 0$ can be chosen arbitrarily small and σ^* depends on d, p, ℓ, α , and γ , as described in (15). Thus, the maximal Besov regularity (w.r.t. the adaptivity scale) $\bar{\sigma}$ of the solution u exceeds its maximal Sobolev regularity \bar{s} provided that $\sigma^* > \bar{s}$. Due to (15), this is the case if α and γ satisfy

$$\ell + \alpha > \bar{s} \quad \text{and} \quad 0 < \gamma < \frac{\ell + \alpha}{d} + \frac{1}{p},$$

or if

$$\frac{d}{d-1} \left(\ell + \alpha + \frac{1}{p} - \gamma \right) > \bar{s} \quad \text{and} \quad \frac{\ell + \alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}. \quad (36)$$

Now the first inequality in (36) is equivalent to $\gamma < \ell + \alpha + 1/p - \bar{s}(d-1)/d$ such that (36) reduces to

$$\frac{\ell + \alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p} - \frac{d-1}{d} \bar{s}.$$

This range for γ is non-empty if and only if $\ell + \alpha > \bar{s}$. In summary, the condition $\ell + \alpha > \bar{s}$ is necessary in both cases and the union of the two ranges for γ yields that $\sigma^* > \bar{s}$ for all values of α and γ satisfying (35), as claimed. \blacksquare

4.2 The p -Poisson equation in two dimensions

As mentioned earlier, in order to derive non-trivial Besov regularity results by means of Theorem 3.1, we need to determine (preferably small) spaces $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ which contain the solutions u to the p -Poisson equation (1); see Subsection 2.1 for the definition of these spaces. For this purpose we proceed as follows. Starting from a known local Hölder regularity result, we estimate the Hölder semi-norms $|u|_{C^{\ell, \alpha}(K)}$ on compact subsets $K \subset\subset \Omega$ in terms of δ_K , in order to conclude estimates on the parameter γ . In what follows we restrict ourselves to the situation $d = 2$, because in this case explicit bounds on the (local) Hölder regularity are available in the literature. In particular, quite recently Lindgren and Lindqvist [35] have proven a lower bound for the Hölder exponent of solutions to (1) with right-hand side $f \in L_q(\Omega)$, $q > 2$; see Proposition 4.11 below.

Remark 4.9. We note in passing that in dimension two we have $L_q(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$, provided that $2/q < 1 + 2/p'$. Hence, Proposition 4.1 guarantees that the problem (1) is uniquely solvable for all $1 < p < \infty$ and $q > 2$.

The subsequent definition is inspired by [35].

Definition 4.10. Let us define the local Hölder exponent $\alpha_q^* = \alpha_q^*(p)$ for $2 < q \leq \infty$ by

-) $1 < p \leq 2$: If $q = \infty$, let α_q^* be any number less than 1, and if $q < \infty$, let

$$\alpha_q^* = 1 - \frac{2}{q}.$$

-) $2 < p < \infty$: If $q = \infty$, let α_q^* be any number less than $1/(p-1)$, and if $q < \infty$, let

$$\alpha_q^* = \frac{1 - 2/q}{p - 1}.$$

The result of Lindgren and Lindqvist [35, Theorem 3] then reads as follows.

Proposition 4.11. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $1 < p < \infty$. For $2 < q \leq \infty$, let $f \in L_q(\Omega)$ and set $\alpha = \alpha_q^*$ as specified in Definition 4.10. Moreover, let $u \in W^1(L_p(\Omega))$ be a solution to (1). Then $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$ and for any compact set $K \subset \Omega$, it holds

$$|u|_{C^{1,\alpha}(K)} \leq C(q, p, \alpha, K) \max\left\{\|f\|_{L_q(\Omega)}^{1/(p-1)}, \|u\|_{L_\infty(\Omega)}\right\}. \quad (37)$$

Remark 4.12. It is known that the Hölder exponent α_q^* defined above is sharp, at least for $p > 2$ and $2 < q \leq \infty$. If $q = \infty$, then this follows from the example given in Remark 4.4. Corresponding examples for finite q can be found in [35].

Based on the local Hölder regularity result given in Proposition 4.11, we are able to show that, for $\alpha = \alpha_q^*$ and certain values of γ , solutions to the p -Poisson equation (1) are contained in locally *weighted* Hölder spaces $C_{\gamma, \text{loc}}^{1,\alpha}(\Omega)$, too; see Proposition 4.14 below. To do so, we have to examine the dependence of the constant $C(q, p, \alpha, K)$ in (37) on $K \subset \subset \Omega$. This is performed in the subsequent lemma.

Lemma 4.13. Let the assumptions of Proposition 4.11 be satisfied. Then, for every disc $B_{r/4} \subset \Omega$ of radius $r/4 > 0$ such that \bar{B}_{2r} is contained in Ω as well, we have

$$|u|_{C^{1,\alpha}(B_{r/4})} \leq C(q, p, \alpha, \Omega) r^{-\alpha-1} \max\left\{\|f\|_{L_q(B_r)}^{1/(p-1)}, \|u\|_{L_\infty(B_r)}\right\} \quad (38)$$

and, for $t > 2$,

$$|u|_{C^{1,\alpha}(B_{r/4})} \leq \hat{C}(q, p, \alpha, \Omega, t) r^{-\alpha-2/t} \max\left\{\|f\|_{L_q(B_{2r})}^{1/(p-1)}, \|\nabla u\|_{L_t(B_{2r})}\right\}. \quad (39)$$

Proof. To show the claim, assume that u solves (1) on the whole domain Ω and let $B_r(x_0) \subset \Omega$ denote a disc of radius $r > 0$ around an arbitrary point x_0 . Then, certainly, u is a solution of the restricted problem $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ in $B_r(x_0)$, as well. Moreover, from Proposition 4.11 we infer that u belongs to $C_{\text{loc}}^{1,\alpha}(\Omega)$ with $\alpha = \alpha_q^*$ given in Definition 4.10. Hence, in particular $u \in L_\infty(B_r(x_0))$.

Now let us perform a translation to the origin. One checks easily that then $\tilde{u} = u(\cdot + x_0)$ solves

$$\operatorname{div}(|\nabla \tilde{u}|^{p-2}\nabla \tilde{u}) = \tilde{f} \quad \text{in } B_r(0),$$

where $\tilde{f} = f(\cdot + x_0)$. Thus, it suffices to prove (38) and (39) only for solutions to the p -Poisson equation (1) in $B_r(0)$, $r > 0$.

To do so, we use a result for the unit disc $B_1(0)$. By Proposition 4.11, with $\bar{\Omega} = B_1(0)$ and $K = B_{1/4}(0)$, we know that if u solves $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ in $B_1(0)$ with $u \in L_\infty(B_1(0))$ and $f \in L_q(B_1(0))$, then there exists a constant $C = C(q, p, \alpha) > 0$, such that for all $x, y \in B_{1/4}(0)$ it holds

$$|\nabla u(x) - \nabla u(y)| \leq C|x - y|^\alpha \max\left\{\|f\|_{L_q(B_1(0))}^{1/(p-1)}, \|u\|_{L_\infty(B_1(0))}\right\}. \quad (40)$$

Now suppose that u solves $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ in some dilated disc $B_r(0)$ and let $F = r^p f(r\cdot)$. Then it is easy to see that $U = u(r\cdot)$ solves

$$\operatorname{div}(|\nabla U|^{p-2}\nabla U) = F \quad \text{in } B_1(0).$$

Clearly, $\|F\|_{L_q(B_1(0))} = r^{p-2/q}\|f\|_{L_q(B_r(0))}$ and $\|U\|_{L_\infty(B_1(0))} = \|u\|_{L_\infty(B_r(0))}$. Next, we apply the estimate (40) to U which yields that for all $x, y \in B_{1/4}(0)$

$$\begin{aligned} & |\nabla u(rx) - \nabla u(ry)| \\ &= r^{-1} |\nabla U(x) - \nabla U(y)| \\ &\leq C r^{-1} |x - y|^\alpha \max\left\{\|F\|_{L_q(B_1(0))}^{1/(p-1)}, \|U\|_{L_\infty(B_1(0))}\right\} \\ &\leq C r^{-1-\alpha} |rx - ry|^\alpha \max\left\{r^{(p-2/q)/(p-1)}\|f\|_{L_q(B_r(0))}^{1/(p-1)}, \|u\|_{L_\infty(B_r(0))}\right\}. \end{aligned}$$

Hence, for all $x \neq y$ in $B_{r/4}(0)$ it holds

$$\begin{aligned} & \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\alpha} \\ &\leq C r^{-1-\alpha} \max\left\{r^{(p-2/q)/(p-1)}\|f\|_{L_q(B_r(0))}^{1/(p-1)}, \|u\|_{L_\infty(B_r(0))}\right\} \\ &\leq \tilde{C} r^{-1-\alpha} \max\left\{\|f\|_{L_q(B_r(0))}^{1/(p-1)}, \|u\|_{L_\infty(B_r(0))}\right\}, \end{aligned} \quad (41)$$

where $\tilde{C} = C \cdot \max\{1, \text{diam}(\Omega)^{(p-2/q)/(p-1)}\}$ and $(p - 2/q)/(p - 1) > 0$, since $2/q < 1 < p$. This shows (38) for all discs $B_{r/4}(0)$ under consideration.

We are left with the proof of (39) for these discs. Note that if u solves (1), so does $u - c$ for every constant c . Hence, from (41) we infer

$$\frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\alpha} \leq C r^{-1-\alpha} \max\left\{r^{(p-2/q)/(p-1)} \|f\|_{L_q(B_r(0))}^{1/(p-1)}, \|u - c\|_{L_\infty(B_r(0))}\right\}, \quad (42)$$

whenever $x \neq y$ belong to $B_{r/4}(0)$. Next we apply Whitney's estimate (see Proposition 5.1) with $k = 1$, $d = 2$, $p = \infty$, and $q = t$. Thus, for every $t > d = 2$ and every square $Q \subset \Omega$, there exist constants c and C' , such that

$$\|u - c\|_{L_\infty(Q)} \leq C' |Q|^{1/2-1/t} \|u\|_{W^1(L_t(Q))}. \quad (43)$$

Let Q_r denote the square in \mathbb{R}^2 with sides parallel to the coordinate axes and side length $2r$ that contains $B_r(0)$. Using the fact that $|Q_r|^{1/2-1/t} = (2r)^{1-2/t}$, from (43) we conclude

$$\|u - c\|_{L_\infty(B_r(0))} \leq C' |Q_r|^{1/2-1/t} \|u\|_{W^1(L_t(Q_r))} \leq C'' r^{1-2/t} \|\nabla u\|_{L_t(B_{2r}(0))} \quad (44)$$

Now, (42) and (44) together yield the upper bound

$$\begin{aligned} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\alpha} &\leq C C'' r^{-2/t-\alpha} \max\left\{r^{-1+2/t+(p-2/q)/(p-1)} \|f\|_{L_q(B_r(0))}^{1/(p-1)}, \|\nabla u\|_{L_t(B_{2r}(0))}\right\}. \end{aligned}$$

Since, clearly,

$$-1 + \frac{2}{t} + \frac{p-2/q}{p-1} = \frac{2}{t} + \frac{1-2/q}{p-1} > 0,$$

by setting $\hat{C} = C \cdot C'' \cdot \max\{1, \text{diam}(\Omega)^{2/t+(1-2/q)/(p-1)}\}$ we finally arrive at

$$\frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\alpha} \leq \hat{C} r^{-2/t-\alpha} \max\left\{\|f\|_{L_q(B_{2r}(0))}^{1/(p-1)}, \|\nabla u\|_{L_t(B_{2r}(0))}\right\}$$

for all $x \neq y$ in $B_{r/4}(0)$. This shows (39) for all discs of interest. ■

The locally weighted Hölder regularity result which forms the basis for our further analysis now can be derived easily from (39):

Proposition 4.14 ($C_{\gamma,\text{loc}}^{1,\alpha}(\Omega)$ regularity). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume $1 < p < \infty$. Furthermore, for $2 < q \leq \infty$ and $f \in L_q(\Omega)$, let $u \in W^1(L_p(\Omega))$ be some solution to the p -Poisson equation (1) and set $\alpha = \alpha_q^*$ as in Definition 4.10.*

(i) *If $|\nabla u| \in L_t(\Omega)$ for some $t > 2$, then we have*

$$u \in C_{\gamma,\text{loc}}^{1,\alpha}(\Omega) \quad \text{for} \quad \alpha = \alpha_q^*, \quad (45)$$

as well as every weight parameter $\gamma \geq \alpha + 2/t$.

(ii) *If $u \in W^s(L_p(\Omega))$ for all $s < \bar{s}$ with some $\bar{s} > \max\{2/p, 1\}$, then (45) holds true for all*

$$\gamma > \alpha + \max\left\{0, 1 - \bar{s} + \frac{2}{p}\right\}.$$

Proof. Let us prove (i). Since the locally weighted Hölder spaces $C_{\gamma,\text{loc}}^{1,\alpha}(\Omega) = C_{\gamma,\text{loc}}^{1,\alpha}(\Omega; \mathcal{K}(c))$ are monotone in γ (see Remark 2.1), we may restrict ourselves to the limiting case $\gamma = \alpha + 2/t$. Moreover, without loss of generality, we can assume $c > 8$; cf. Section 2. Then let us consider a compact disc $B_r \in \mathcal{K}(c)$, i.e., $B_r = B_r(x_0)$ with $x_0 \in \Omega$ and $r > 0$ such that the (open) disc $\mathring{B}_{cr}(x_0)$ still is contained in Ω . Clearly, $r < \text{dist}(x_0, \partial\Omega)/8$, so that we can choose $R \geq r$ with

$$\frac{\text{dist}(x_0, \partial\Omega)}{16} < R < \frac{\text{dist}(x_0, \partial\Omega)}{8}.$$

Consequently, $B_R = B_R(x_0)$ is a compact disc with $B_r \subseteq B_R \subset \mathring{B}_{8R} \subset \Omega$. Therefore, (39) applied for B_R yields

$$|u|_{C^{1,\alpha}(B_r)} \leq |u|_{C^{1,\alpha}(B_R)} \leq C R^{-\alpha-2/t} \max\left\{\|f|_{L_q(B_{8R})}\|^{1/(p-1)}, \|\nabla u|_{L_t(B_{8R})}\|\right\},$$

where $C = C(q, p, \alpha, \Omega)$ does not depend on r . Since $\delta_{B_r} < \text{dist}(x_0, \partial\Omega) < 16R$ and $\gamma = \alpha + 2/t$, setting $C' = C \cdot 16^\gamma$ we may estimate further

$$|u|_{C^{1,\alpha}(B_r)} \leq C' \delta_{B_r}^{-\gamma} \max\left\{\|f|_{L_q(\Omega)}\|^{1/(p-1)}, \|\nabla u|_{L_t(\Omega)}\|\right\}.$$

Observe that the latter maximum is finite due to the additional assumption that $|\nabla u|$ belongs to $L_t(\Omega)$. Multiplying by $\delta_{B_r}^\gamma$ and taking the supremum over all $B_r \in \mathcal{K}(c)$ thus proves the claim stated in (i).

The proof of (ii) follows from Sobolev's embedding: At first, note that $\bar{s} > 2/p$ yields that $1 > \max\{0, 1 - \bar{s} + 2/p\}$. Therefore, we can choose $s < \bar{s}$ and $t > 2$ such that $2/t > \max\{0, 1 - s + 2/p\}$ is arbitrary close to $\max\{0, 1 - \bar{s} + 2/p\}$. Thus, in view of (45), it remains to show that $|\nabla u| \in L_t(\Omega)$ for this choice of s and t . To do so, observe that

$s - 1 > 2/p - 2/t$. Since we imposed the additional condition that $\bar{s} > 1$, we may assume that $s - 1 > 0$. Hence, it follows

$$s - 1 > 2 \cdot \max \left\{ 0, \frac{1}{p} - \frac{1}{t} \right\}$$

which particularly implies the embedding $W^{s-1}(L_p(\Omega)) \hookrightarrow L_t(\Omega)$. Finally, the fact that $u \in W^s(L_p(\Omega))$ yields $|\nabla u| \in W^{s-1}(L_p(\Omega))$ completes the proof. \blacksquare

Next let us combine the locally weighted Hölder regularity result obtained in Proposition 4.14 above with the generic Besov regularity result stated in Theorem 4.8. This leads to conditions on the Sobolev smoothness of solutions u to the p -Poisson equation (1) which imply (non-trivial) Besov regularity assertions for these u .

Theorem 4.15. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume $1 < p < \infty$. Moreover, for $2 < q \leq \infty$, as well as $f \in L_q(\Omega)$, let u be some solution to the p -Poisson equation (1) which satisfies $u \in W^s(L_p(\Omega))$ for all $s < \bar{s}$. Then the conditions*

-) $1 < p \leq 2$ and $\frac{2}{p} < \bar{s} < 2 - \frac{2}{q}$,
-) $2 < p < \infty$ and $1 < \bar{s} < 1 + \frac{1-2/q}{p-1}$

imply that there exists $\bar{\sigma} > \bar{s}$ such that

$$u \in B_\tau^\sigma(L_\tau(\Omega)) \quad \text{for all} \quad 0 < \sigma < \bar{\sigma} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}. \quad (46)$$

Proof. Note that our assumptions particularly imply

$$\max \left\{ 1, \frac{2}{p} \right\} < \bar{s} < 2. \quad (47)$$

Therefore, in view of Theorem 4.8 (applied with $d = 2$ and $\ell = 1$), it suffices to find parameters α and γ with $\bar{s} - 1 < \alpha \leq 1$ and

$$0 < \gamma < 1 + \alpha + \frac{1}{p} - \frac{\bar{s}}{2} \quad (48)$$

such that $u \in C_{\gamma, \text{loc}}^{1, \alpha}(\Omega)$. Observe that from (47) it follows

$$\alpha + \max \left\{ 0, 1 - \bar{s} + \frac{2}{p} \right\} < 1 + \alpha + \frac{1}{p} - \frac{\bar{s}}{2} \quad \text{for all} \quad 0 < \alpha \leq 1.$$

Thus, due to Proposition 4.14(ii), choosing $\alpha = \alpha_q^*$ (as given in Definition 4.10), there exists γ which satisfies (48) such that $u \in C_{\gamma, \text{loc}}^{1, \alpha}(\Omega)$. To complete the proof, it remains to check that this choice of α belongs to the interval $(\bar{s} - 1, 1]$ which is obvious in view of Definition 4.10, as well as our restrictions on \bar{s} . \blacksquare

Remark 4.16. Note that the bound \bar{s} in Theorem 4.15 can be calculated explicitly, provided that the maximal Sobolev regularity \bar{s} is known; see, e.g., the proof of Theorem 4.17 below.

Now we are well-prepared to state and prove one of the main results of this paper. It shows that for a large range of parameters p and q the (unique) solution to (28), i.e., to the p -Poisson with homogeneous Dirichlet boundary conditions, has a significantly higher Besov regularity compared to its Sobolev smoothness. Indeed, as we shall see, on bounded Lipschitz domains $\Omega \subset \mathbb{R}^2$ this happens whenever $4/3 < p < \infty$ and $\max\{4, 2p\} < q \leq \infty$. Therefore, for the same range of parameters, the application of adaptive (wavelet) algorithms for the numerical treatment of (28) is completely justified. Recall that from Proposition 4.5 (and the subsequent remarks) it follows that the solution u to this problem is contained in $W^s(L_p(\Omega))$ for all $s < s^*$ given in (30). Consequently, the proof of the subsequent result is obtained by applying Theorem 4.15 with $\bar{s} = s^*$ together with some straightforward calculations.

Theorem 4.17 (Besov regularity on Lipschitz domains in 2D). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $1 < p < \infty$, as well as $f \in L_q(\Omega)$ with $2 < q \leq \infty$ and $q \geq p'$. Then the unique solution u to the p -Poisson equation with homogeneous Dirichlet boundary conditions (28) satisfies*

$$u \in B_\tau^\sigma(L_\tau(\Omega)) \quad \text{for all} \quad 0 < \sigma < \bar{\sigma} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p},$$

where

$$\bar{\sigma} = \begin{cases} \frac{3}{2} & \text{if } 1 < p < 4/3 \text{ and } p' \leq q \leq \infty, \\ \frac{3}{2} & \text{if } p = 4/3 \text{ and } 4 < q \leq \infty, \\ 3 - \frac{2}{p} & \text{if } 4/3 < p \leq 2 \text{ and } (\frac{1}{p} - \frac{1}{2})^{-1} \leq q \leq \infty, \\ 2 - \frac{2}{q} & \text{if } 4/3 < p \leq 2 \text{ and } 4 < q < (\frac{1}{p} - \frac{1}{2})^{-1}, \\ \frac{3}{2} & \text{if } 4/3 \leq p < 2 \text{ and } p' \leq q \leq 4, \\ \frac{3}{2} & \text{if } p = 2 \text{ and } 2 < q \leq 4, \\ 1 + \frac{1-2/q}{p-1} & \text{if } 2 < p < \infty \text{ and } 2p < q \leq \infty, \\ 1 + \frac{1}{p} & \text{if } 2 < p < \infty \text{ and } 2 < q \leq 2p. \end{cases}$$

Proof. Step 1. Let us start with the cases where $\bar{\sigma} = s^*$, i.e., where $\bar{\sigma}$ equals $3/2$ or $1 + 1/p$. Then from classical embeddings of Besov spaces it follows that $u \in W^s(L_p(\Omega))$ for all $0 < s < \bar{\sigma}$ implies that u also belongs to $B_p^s(L_p(\Omega))$ for all these s which in turn yields the claim; cf. Remark 2.2.

Step 2. We are left with proving the assertion for the third, fourth, and seventh line in the definition of $\bar{\sigma}$. According to (the proof of) Theorem 4.15 we know that in all these remaining cases Proposition 4.14(ii) ensures the existence of some reasonably small γ such that $u \in C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$, where $\alpha = \alpha_q^*$ (as given in Definition 4.10) and $\ell = 1$. In fact, it can be checked that we can use

$$\gamma = \alpha + \varepsilon + \begin{cases} 2/p - 1/2 & \text{if } p < 2, \\ 1/p, & \text{if } p \geq 2 \end{cases}$$

with arbitrarily small $\varepsilon > 0$. As shown in the proof of Theorem 4.8 (which we used to derive Theorem 4.15), the desired quantity $\bar{\sigma}$ then is given by σ^* defined in (15) in Theorem 3.1. Thus, we need to determine whether our choice of γ is smaller or larger than $(1 + \alpha)/2 + 1/p$. Note that, according to Theorem 4.8, we already know that for all cases of interest it is smaller than $1 + \alpha + 1/p$. It turns out that for $4/3 < p \leq 2$ and $(1/p - 1/2)^{-1} \leq q \leq \infty$, i.e., for the constellation described in the third line, the second case in (15) applies, i.e., then

$$\frac{1 + \alpha}{2} + \frac{1}{p} \leq \gamma < 1 + \alpha + \frac{1}{p}.$$

Consequently, for these p and q , the quantity $\bar{\sigma} = \sigma^*$ is given by $2(1 + \alpha + 1/p - \gamma) = 3 - 2/p - \varepsilon$, where ε can be neglected since it can be chosen arbitrarily small.

For the remaining two ranges for p and q the chosen weight γ is small enough such that the first case in (15) applies. Thus, for p and q as described in the fourth and seventh line, we obtain $\bar{\sigma} = \sigma^* = \ell + \alpha$ with $\ell = 1$ and $\alpha = \alpha_q^*$. This finishes the proof. \blacksquare

In the more restrictive (but practically more important) setting of polygonal domains slightly better Besov regularity assertions for the unique solutions to (28) with $f \in L_q(\Omega)$ can be deduced using our method, at least for some cases. For this purpose, we will employ a further Sobolev regularity result which was shown by Ebmeyer [23, Corollary 2.3] for polyhedral Lipschitz domains in arbitrary dimensions:

Proposition 4.18. *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral Lipschitz domain and for $1 < p < \infty$ let $f \in L_{p'}(\Omega)$. Then the unique solution $u \in W^1(L_p(\Omega))$ to (28) satisfies*

$$|\nabla u| \in L_t(\Omega) \quad \text{for all} \quad t < \frac{d}{d-1} p.$$

Remark 4.19. The example described in Remark 4.7 shows that, for $d = 2$, Ebmeier's result (Proposition 4.18) is sharp, meaning that there are cases in which

$$|\nabla u| \notin L_t(\Omega) \quad \text{if} \quad t > 2p = \frac{d}{d-1} p.$$

Our improved Besov regularity result for solutions to p -Poisson equations with homogeneous boundary conditions (28) on bounded polygonal domains then reads as follows.

Theorem 4.20 (Besov regularity on polygonal domains). *Let $\Omega \subset \mathbb{R}^2$ denote a bounded polygonal domain and let $1 < p < \infty$, as well as $f \in L_q(\Omega)$ with $2 < q \leq \infty$ and $q \geq p'$. Then the unique solution u to the p -Poisson equation with homogeneous Dirichlet boundary conditions (28) satisfies*

$$u \in B_\tau^\sigma(L_\tau(\Omega)) \quad \text{for all} \quad 0 < \sigma < \bar{\sigma} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p},$$

where

$$\bar{\sigma} = \begin{cases} 2 - \frac{2}{q} & \text{if } 1 < p < 4/3 \text{ and } p' \leq q \leq \infty, \\ 2 - \frac{2}{q} & \text{if } p = 4/3 \text{ and } 4 < q \leq \infty, \\ 2 - \frac{2}{q} & \text{if } 4/3 < p \leq 2 \text{ and } 4 < q \leq \infty, \\ \frac{3}{2} & \text{if } 4/3 \leq p < 2 \text{ and } p' \leq q \leq 4, \\ \frac{3}{2} & \text{if } p = 2 \text{ and } 2 < q \leq 4, \\ 1 + \frac{1-2/q}{p-1} & \text{if } 2 < p < \infty \text{ and } 2p < q \leq \infty, \\ 1 + \frac{1}{p} & \text{if } 2 < p < \infty \text{ and } 2 < q \leq 2p. \end{cases}$$

Before giving the proof of this assertion we want to stress that in the first three cases, as well as in the sixth one, the upper bound $\bar{\sigma}$ for the regularity of the solution u in the adaptivity scale of Besov spaces is strictly larger than $\bar{s} = s^*$ as defined in (30) which is considered to be a sharp bound for the regularity in the Sobolev scale; see Remark 4.7. Hence, in contrast to Theorem 4.17 (which deals with general bounded Lipschitz domains in \mathbb{R}^2), on polygonal domains u gains some additional regularity also in the range $1 < p \leq 4/3$ (except for the case $p = 4/3$ and $q = 4$). Furthermore, observe that for the case of $p \in (4/3, 2)$ and large q the value $3 - 2/p$ for Lipschitz domains is strictly worse than $2 - 2/q$ obtained in Theorem 4.20 for polygonal domains. Finally we note that, given some fixed p , in all cases in which $\bar{\sigma} > \bar{s}$ this quantity grows with increasing integrability q of the right-hand side f . This is not the case for s^* . Accordingly, the largest gain $\bar{\sigma} - \bar{s}$ is obtained for $f \in L_\infty(\Omega)$. This situation is illustrated in Figure 1 below.

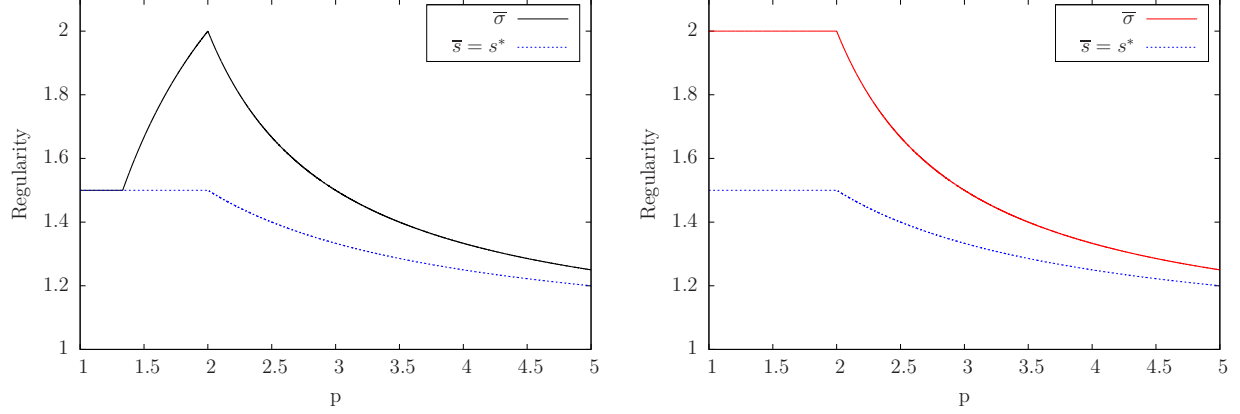


Figure 1: Bounds $\bar{\sigma}$ and s^* for the regularity of solutions u to (28) with $f \in L_\infty(\Omega)$ on bounded 2D Lipschitz domains (left) and bounded polygonal domains (right), measured in $B_\tau^\sigma(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, and in $W^s(L_p(\Omega))$, respectively.

Proof (of Theorem 4.20). Step 1. Since $q \geq p'$, we have that $L_q(\Omega) \hookrightarrow L_{p'}(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$. Consequently, Proposition 4.1 assures a unique solution $u \in W^1(L_p(\Omega))$. Then Remark 2.2(iv) implies $u \in B_p^{1-\varepsilon}(L_p(\Omega))$ for all $\varepsilon \in (0, 1)$. Moreover, by Proposition 4.14(i) we know that $u \in C_{\gamma, \text{loc}}^{1, \alpha}(\Omega)$ for all $\gamma \geq \alpha + 2/t$, with $\alpha = \alpha_q^*$ given in Definition 4.10 and $t > 2$ such that $|\nabla u| \in L_t(\Omega)$. Proposition 4.18 shows that the latter condition is fulfilled for all $t < 2p$, i.e., for all $2/t$ strictly larger (but arbitrary close to) $1/p$. Thus, since $\alpha \in (0, 1)$, we can choose γ such that

$$\alpha + \frac{1}{p} < \gamma < \frac{1 + \alpha}{2} + \frac{1}{p}.$$

Then, for this choice of α and γ , as well as $d = 2$, $s = 1 - \varepsilon$, and $\ell = 1$, we apply Theorem 3.1 (note that every polygonal domain $\Omega \subset \mathbb{R}^2$ is Lipschitz!) and conclude that u belongs to $B_\tau^\sigma(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, for all

$$0 < \sigma < \min \left\{ 1 + \alpha, \frac{2}{2 - 1} (1 - \varepsilon) \right\} = 1 + \alpha,$$

where the last equality holds provided that $\varepsilon > 0$ is chosen sufficiently small.

Step 2. Since $f \in L_{p'}(\Omega)$, we furthermore can employ Proposition 4.5 (as well as the subsequent remarks) to see that $u \in W^s(L_p(\Omega))$ for all $s < s^*$. This implies that u belongs to $B_p^s(L_p(\Omega))$ and $B_\tau^\sigma(L_\tau(\Omega))$ for all s and σ less than s^* , respectively.

In conclusion, combining both steps yields

$$u \in B_\tau^\sigma(L_\tau(\Omega)) \quad \text{for all} \quad 0 < \sigma < \max\{1 + \alpha, s^*\} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}.$$

Now the claim directly follows from the definitions of $\alpha = \alpha_q^*$ and s^* . ■

Remark 4.21. We add some comments on our main results in Theorem 4.17 and 4.20, resp.:

- (i) The restriction $q \geq p'$ in Theorem 4.17 can be weakened. Anyhow, note that for p in the vicinity of 1 and q close to 2, Proposition 4.5 only guarantees that the unique solution u to (28) satisfies $u \in W^s(L_p(\Omega))$ for all $s < \bar{s}$ with some $1 \leq \bar{s} < s^*$.
- (ii) According to [23, Section 5.3] Proposition 4.18 remains valid for special classes of bounded Lipschitz domains with polyhedral structure. Hence, also Theorem 4.20 applies to this slightly generalized situation.
- (iii) Observe that for large q our bound $\bar{\sigma}$ in Theorem 4.20 always equals $1 + \alpha$, where $\alpha = \alpha_q^*$ is the local Hölder exponent given in Definition 4.10 which is known to be optimal at least for $p > 2$; see Remark 4.12. Thus, by (15), as well as the subsequent statements, we see that the results stated in Theorem 4.20 are the best possible we can achieve by our method (i.e., by Theorem 3.1). On the other hand, we do not know whether they are sharp, as (for general p) in the current literature there seem to exist no results at all which address comparable regularity questions. However, for example in the case of the classical Laplacian ($p = 2$) Besov regularity larger than two cannot be expected for general right-hand sides of smoothness zero, since then we deal with a linear operator of order two.

Finally, let us briefly consider p -harmonic functions, i.e., solutions to the p -Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } \Omega, \quad (49)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and $1 < p < \infty$. In [23, Remark 2.5(iv)] Ebmeyer states that if Ω is a bounded polyhedral Lipschitz domain (of arbitrary dimension $d \geq 2$), then all solutions to (49) with boundary data $g \in W^1(L_p(\partial\Omega))$ are as well contained in $W^s(L_p(\Omega))$ for all $s < s^*$ defined by (30). However, he does not provide a proof of this statement. Using this claim, the arguments in Step 1 of the proof of Theorem 4.20 would imply that all p -harmonic functions u on bounded polygonal domains Ω satisfy

$$u \in B_\tau^\sigma(L_\tau(\Omega)) \quad \text{for all} \quad 0 < \sigma < \begin{cases} 2 & \text{if } 1 < p \leq 2, \\ 1 + \frac{1}{p-1} & \text{if } 2 < p < \infty \end{cases} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}. \quad (50)$$

In addition, we remark that the local Hölder regularity of two-dimensional p -harmonic functions is known to be higher than for general solutions to the p -Poisson equation (1): In fact, Iwaniec and Manfredi [29] showed that in the case $d = 2$ all p -harmonic functions are contained in $C_{\text{loc}}^{\ell, \alpha}(\Omega)$, where $\ell \in \mathbb{N}$ and $0 < \alpha \leq 1$ are determined by the formula

$$\ell + \alpha = 1 + \frac{1}{6} \left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}} \right). \quad (51)$$

Furthermore, for $p \neq 2$ this result is known to be sharp; see [29]. Note that for all $1 < p < \infty$ the right-hand side of (51) indeed is larger than $1 + \alpha_\infty^*$. In conclusion, one might expect to achieve even higher Besov regularity for p -harmonic functions than stated in (50). To prove this conjecture (by means of our embedding result Theorem 3.1), we would need to exploit the sharp Hölder regularity (51) instead of Proposition 4.11; provided we could show that p -harmonic functions belong to $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ for these ℓ and α , as well as for sufficiently small values of γ , and provided that Ebmeyer's claim holds true. Unfortunately, sufficient estimates for the parameter γ do not seem to exist, yet.

5 Appendix

This final part of the paper is concerned with estimates needed in our proofs, as well as with auxiliary assertions that are of interest on their own.

To begin with, we state the following well-known Whitney-type estimates which can be found, e.g., in DeVore [14, Subsection 6.1]. Here and in what follows we let $\Pi_k(S)$ denote the set of all polynomials \mathcal{P} on some bounded and simply connected set $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, which possess a total degree $\deg \mathcal{P}$ not larger than $k \in \mathbb{N}_0$. As usual, $\lceil x \rceil$ (and $\lfloor x \rfloor$, respectively) means the smallest (largest) integer larger (smaller) or equal to $x \in \mathbb{R}$.

Proposition 5.1 (Whitney's estimate). *For $d \in \mathbb{N}$ let Q denote an arbitrary cube in \mathbb{R}^d with sides parallel to the coordinate axes. Moreover,*

(i) *let $1 \leq p, q \leq \infty$ and $k \in \mathbb{N}$ with $k > d \max\{0, 1/q - 1/p\}$. Then it holds*

$$\inf_{\mathcal{P} \in \Pi_{k-1}(Q)} \|f - \mathcal{P}\|_{L_p(Q)} \leq C |Q|^{k/d+1/p-1/q} |f|_{W^k(L_q(Q))},$$

whenever the right-hand side is finite. Therein the constant C depends only on k .

(ii) *let $1 \leq p \leq \infty$ and $0 < q \leq \infty$. Furthermore, assume that $0 < t < \infty$ satisfies $t \geq d \max\{0, 1/q - 1/p\}$. Then we have*

$$\inf_{\mathcal{P} \in \Pi_{\lceil t \rceil - 1}(Q)} \|f - \mathcal{P}\|_{L_p(Q)} \leq C |Q|^{t/d+1/p-1/q} |f|_{B_q^t(L_q(Q))},$$

whenever the right-hand side is finite. Here the constant C depends only on t .

In the proof of our general embedding result (Theorem 3.1) the subsequent bound is used. As no explicit derivation of this quite natural assertion seems to be available in the literature, a detailed proof is added here for the reader's convenience.

Proposition 5.2. *For $d \in \mathbb{N}$ let Q denote some open cube in \mathbb{R}^d with sides parallel to the coordinate axes. Then for all $\ell \in \mathbb{N}_0$ and $0 < \alpha \leq 1$ it holds*

$$|g|_{B_{\infty}^{\ell+\alpha}(L_{\infty}(Q))} \lesssim |g|_{C^{\ell,\alpha}(Q)},$$

whenever the right-hand side is finite.

Proof. Step 1. Assume that $\ell = 0$. Then, for $0 < \alpha < 1$, the assertion follows from the definition of the involved semi-norms; see (4) and (6) in Section 2. If $\alpha = 1$, then we use the triangle inequality to see that for all $h \in \mathbb{R}^d$ it holds

$$\left\| \Delta_h^2(g, \cdot) \right\|_{L_{\infty}(Q_{2,h})} = \left\| \Delta_h^1(g, \cdot + h) - \Delta_h^1(g, \cdot) \right\|_{L_{\infty}(Q_{2,h})} \lesssim \left\| \Delta_h^1(g, \cdot) \right\|_{L_{\infty}(Q_{1,h})}, \quad (52)$$

where we recall that for $r \in \mathbb{N}$ the set $Q_{r,h}$ denotes the collection of all $x \in Q$ such that $[x, x + rh] \subset Q$. Then, as before, the claim directly follows from the definitions of the semi-norms.

Step 2. Now let $\ell \in \mathbb{N}$. Given $t > 0$, as well as $h \in \mathbb{R}^d$ with $0 < |h| \leq t$, and any function f on some domain $\Omega \subset \mathbb{R}^d$, the mean value theorem ensures that for all $x \in \Omega_{1,h}$ there exists some $\xi_x \in [x, x + h] \subset \Omega$ with

$$\left| \Delta_h^1(f, x) \right| = |h \cdot \nabla f(\xi_x)| \leq |h| |\nabla f(\xi_x)| \lesssim t \sum_{|\nu|=1} |\partial^{\nu} f(\xi_x)|,$$

whenever the right-hand side is finite. Obviously, the same is true also for $h = 0$. Thus, we conclude that for every such f and all $|h| \leq t$

$$\left\| \Delta_h^1(f, \cdot) \right\|_{L_{\infty}(\Omega_{1,h})} \lesssim t \sup_{x \in \Omega_{1,h}} \sum_{|\nu|=1} |\partial^{\nu} f(\xi_x)| \leq t \sum_{|\nu|=1} \|\partial^{\nu} f\|_{L_{\infty}(\Omega)}. \quad (53)$$

Observe that $r := \lfloor \ell + \alpha \rfloor + 1 \geq 2$ for all $0 < \alpha \leq 1$. Therefore, if we use (53) for $f := \Delta_h^{r-1}(g, \star)$ together with the linearity of ∂^{ν} and Δ_h^{r-1} ,

$$\begin{aligned} |g|_{B_{\infty}^{\ell+\alpha}(L_{\infty}(Q))} &= \sup_{t>0} t^{-(\ell+\alpha)} \sup_{h \in \mathbb{R}^d, |h| \leq t} \left\| \Delta_h^1(\Delta_h^{r-1}(g, \star), \cdot) \right\|_{L_{\infty}(Q_{r,h})} \\ &\lesssim \sup_{t>0} t^{-(\ell+\alpha)} \sup_{h \in \mathbb{R}^d, |h| \leq t} t \sum_{|\nu|=1} \left\| \partial^{\nu} \Delta_h^{r-1}(g, \star) \right\|_{L_{\infty}(\Omega_{r-1,h})} \\ &\leq \sum_{|\nu|=1} \sup_{t>0} t^{-(\ell+\alpha)+1} \sup_{h \in \mathbb{R}^d, |h| \leq t} \left\| \Delta_h^{r-1}(\partial^{\nu} g, \cdot) \right\|_{L_{\infty}(\Omega_{r-1,h})}. \end{aligned}$$

If necessary, we can iterate this argument and deduce

$$|g|_{B_{\infty}^{\ell+\alpha}(L_{\infty}(Q))} \lesssim \sum_{|\nu|=r-1} \sup_{t>0} t^{-(\ell+\alpha)+r-1} \sup_{h \in \mathbb{R}^d, |h| \leq t} \left\| \Delta_h^1(\partial^{\nu} g, \cdot) \right\|_{L_{\infty}(\Omega_{1,h})}. \quad (54)$$

For $0 < \alpha < 1$ it is $r - 1 = \ell$. Consequently, in this case we obtain

$$\begin{aligned} |g|_{B_{\infty}^{\ell+\alpha}(L_{\infty}(Q))} &\lesssim \sum_{|\nu|=\ell} \sup_{t>0} \sup_{h \in \mathbb{R}^d, |h| \leq t} \frac{\|\partial^{\nu} g(\cdot + h) - \partial^{\nu} g(\cdot)\|_{L_{\infty}(\Omega_{1,h})}}{t^{\alpha}} \\ &= \sum_{|\nu|=\ell} \sup_{\substack{x, y \in Q, \\ x \neq y}} \frac{|\partial^{\nu} g(x) - \partial^{\nu} g(y)|}{|x - y|^{\alpha}}. \end{aligned} \quad (55)$$

Since the last term equals $|g|_{C^{\ell, \alpha}(Q)}$, this shows the claim in the case $\alpha < 1$.

Finally, we note that if $\alpha = 1$, then $r \geq 3$. Thus, by means of the same (iterative) argument as above, this time we derive

$$|g|_{B_{\infty}^{\ell+\alpha}(L_{\infty}(Q))} \lesssim \sum_{|\nu|=r-2} \sup_{t>0} t^{-(\ell+\alpha)+r-2} \sup_{h \in \mathbb{R}^d, |h| \leq t} \left\| \Delta_h^2(\partial^{\nu} g, \cdot) \right\|_{L_{\infty}(\Omega_{2,h})}$$

instead of (54). Using $r - 2 = \ell$ in conjunction with an estimate similar to (52) from Step 1 this allows to conclude (55) also for this case. Hence, the proof is complete. \blacksquare

In Remark 2.1, among other things, we stated that intersections of locally weighted Hölder spaces (as introduced in Subsection 2.1) with certain Besov spaces form Banach spaces w.r.t. the canonical maximum norm. Proposition 5.3 below is devoted to this claim. The subsequent three lemmata are used to derive a sound mathematical proof.

Proposition 5.3. *For $d \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and for $\ell \in \mathbb{N}_0$, $0 < \alpha \leq 1$, as well as $\gamma > 0$, let $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ denote a locally weighted Hölder space. Then for all $s > 0$ and $1 \leq p, q \leq \infty$ the space*

$$B_q^s(L_p(\Omega)) \cap C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega) \quad (56)$$

endowed with the norm

$$\|\cdot\| = \max \left\{ \left\| \cdot \right\|_{B_q^s(L_p(\Omega))}, \left\| \cdot \right\|_{C_{\gamma, \text{loc}}^{\ell, \alpha}} \right\} \quad (57)$$

is a Banach space.

Proof. Since $\|\cdot\|_{B_q^s(L_p(\Omega))}$ is a norm on $B_q^s(L_p(\Omega))$ and $|\cdot|_{C_{\gamma,\text{loc}}^{\ell,\alpha}}$ defines a semi-norm for $C_{\gamma,\text{loc}}^{\ell,\alpha}(\Omega)$, it obviously holds that $\|\cdot\|$ is a norm for the space (56). To show completeness, let $\{f_j\}_{j \in \mathbb{N}_0}$ be a Cauchy sequence in (56) with respect to $\|\cdot\|$. Then, by completeness of the Besov space, there exists some $f \in B_q^s(L_p(\Omega))$ such that

$$f_j \rightarrow f \quad \text{in } B_q^s(L_p(\Omega)), \quad \text{as } j \rightarrow \infty. \quad (58)$$

This clearly remains true for all restrictions of f_j and f , respectively, e.g., when Ω is replaced by an open ball $\tilde{B} \subset \Omega$.

In the following, we will show that f_j converges to f with respect to $|\cdot|_{C_{\gamma,\text{loc}}^{\ell,\alpha}}$, too. Let $B = B_r(x_0) \subset \Omega$ be a non-empty closed ball such that $B_{cr}(x_0)$ is still contained in Ω for some $c > 1$. Given some function $g \in C^\ell(B)$ we denote by $T^{\ell,x_0}[g]$ its Taylor polynomial of degree ℓ at x_0 , i.e.,

$$T^{\ell,x_0}[g](x) = \sum_{|\nu| \leq \ell} \frac{\partial^\nu g(x_0)}{\nu!} (x - x_0)^\nu, \quad x \in B.$$

Step 1. Here we prove that, if $\{f_j\}_{j \in \mathbb{N}_0}$ is a Cauchy sequence w.r.t. $|\cdot|_{C^{\ell,\alpha}(B)}$, then

$$\{f_j - T^{\ell,x_0}[f_j]\}_{j \in \mathbb{N}_0} \quad (59)$$

forms a Cauchy sequence with respect to the norm in the Hölder space $C^{\ell,\alpha}(B)$,

$$\|\cdot\|_{C^{\ell,\alpha}(B)} = \|\cdot\|_{C^\ell(B)} + |\cdot|_{C^{\ell,\alpha}(B)}.$$

Since the definition of the semi-norm $|\cdot|_{C^{\ell,\alpha}(B)}$ given in (4) is based on derivatives of degree ℓ , we have

$$|f_j - T^{\ell,x_0}[f_j]|_{C^{\ell,\alpha}(B)} = |f_j|_{C^{\ell,\alpha}(B)}. \quad (60)$$

Therefore it remains to show that (59) is a Cauchy sequence with respect to the norm $\|\cdot\|_{C^\ell(B)}$. For $j, k \in \mathbb{N}_0$ let $g_{j,k} = f_j - f_k$ and choose $\nu \in \mathbb{N}_0^d$ with $|\nu| \leq \ell$. Then, by linearity of the Taylor polynomial, for all $x \in B$ it holds

$$\begin{aligned} \partial^\nu \left((f_j - T^{\ell,x_0}[f_j]) - (f_k - T^{\ell,x_0}[f_k]) \right)(x) &= \partial^\nu (g_{j,k} - T^{\ell,x_0}[g_{j,k}])(x) \\ &= \partial^\nu g_{j,k}(x) - T^{\ell-|\nu|,x_0}[\partial^\nu g_{j,k}](x). \end{aligned} \quad (61)$$

According to Lemma 5.5 below, we thus have

$$\sup_{x \in B} |\partial^\nu g_{j,k}(x) - T^{\ell-|\nu|,x_0}[\partial^\nu g_{j,k}](x)| \lesssim |\partial^\nu g_{j,k}|_{C^{\ell-|\nu|,\alpha}(B)} \leq |f_j - f_k|_{C^{\ell,\alpha}(B)}$$

for all $|\nu| \leq \ell$. Together with (61) this shows that

$$\left\| \left(f_j - T^{\ell, x_0}[f_j] \right) - \left(f_k - T^{\ell, x_0}[f_k] \right) \right\|_{C^\ell(B)} \lesssim |f_j - f_k|_{C^{\ell, \alpha}(B)},$$

i.e., (59) forms a Cauchy sequence w.r.t. $\left\| \cdot \right\|_{C^\ell(B)}$. This observation in conjunction with (60) finally proves that $\{f_j - T^{\ell, x_0}[f_j]\}_{j \in \mathbb{N}_0}$ is a Cauchy sequence in the norm of the Hölder space $C^{\ell, \alpha}(B)$, too.

Step 2. Of course, the space $C^{\ell, \alpha}(B)$ endowed with the norm $\left\| \cdot \right\|_{C^{\ell, \alpha}(B)}$ is complete. Since we have shown that $\{f_j - T^{\ell, x_0}[f_j]\}_{j \in \mathbb{N}_0}$ is a Cauchy sequence with respect to this norm, there exists some $f_B \in C^{\ell, \alpha}(B)$ such that

$$\left(f_j - T^{\ell, x_0}[f_j] \right) \rightarrow f_B \quad \text{in} \quad \left\| \cdot \right\|_{C^{\ell, \alpha}(B)}, \quad \text{as } j \rightarrow \infty.$$

Step 3. In the previous steps it was proven that every Cauchy sequence $\{f_j\}_{j \in \mathbb{N}_0}$ in $B_q^s(L_p(\Omega)) \cap C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ (w.r.t. $\left\| \cdot \right\|$) converges to some f in $B_q^s(L_p(\Omega))$ and that for every non-empty closed ball $B = B_r(x_0) \subset \mathbb{R}^d$ for which $B_{cr}(x_0)$ is still contained in Ω the sequence $\{f_j - T^{\ell, x_0}[f_j]\}_{j \in \mathbb{N}_0}$ restricted to B converges to some f_B with respect to $\left\| \cdot \right\|_{C^{\ell, \alpha}(B)}$. It remains to show that $f_j \rightarrow f$ in the semi-norm of $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$. Let \mathring{B} be the interior of B . Lemma 5.6, applied to $X = B_q^s(L_p(\mathring{B}))$, implies that the restriction of f to B belongs to $C^{\ell, \alpha}(B)$ and that

$$f_j \rightarrow f \quad \text{with respect to} \quad \left| \cdot \right|_{C^{\ell, \alpha}(B)}, \quad \text{as } j \rightarrow \infty.$$

Since clearly, for all $j \in \mathbb{N}_0$ and every B , it holds

$$|f_j - f|_{C^{\ell, \alpha}(B)} \leq \lim_{k \rightarrow \infty} |f_j - f_k|_{C^{\ell, \alpha}(B)},$$

the definition of $\left| \cdot \right|_{C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)}$ as a weighted supremum of $C^{\ell, \alpha}(B)$ -semi-norms yields

$$|f_j - f|_{C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)} \leq \lim_{k \rightarrow \infty} |f_j - f_k|_{C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)}.$$

Hence, from the assumption that $\{f_j\}_{j \in \mathbb{N}_0}$ is a Cauchy sequence in $C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega)$ and by (58) it follows that

$$f_j \rightarrow f \quad \text{in} \quad B_q^s(L_p(\Omega)) \cap C_{\gamma, \text{loc}}^{\ell, \alpha}(\Omega), \quad \text{as } j \rightarrow \infty,$$

and thus the proof is finished. ■

Remark 5.4. Let $s > 0$. If $0 < p < 1$ or $0 < q < 1$, then $B_q^s(L_p(\Omega))$ is only a quasi-Banach space, i.e., it is complete with respect to the *quasi-norm* $\|\cdot\|_{B_q^s(L_p(\Omega))}$. However, in the same way as in Proposition 5.3, one can show that in this case the intersection (56) endowed with the quasi-norm $\|\cdot\|$ given by (57) defines a quasi-Banach space.

Lemma 5.5. Let $B \subset \mathbb{R}^d$, $d \in \mathbb{N}$, denote a non-trivial closed ball with center x_0 and let $\ell \in \mathbb{N}_0$. For $g \in C^{\ell, \alpha}(B)$ let $T^{\ell, x_0}[g]$ be the Taylor polynomial of degree ℓ at x_0 . Then there exists a constant $C_{\ell, \alpha, B} > 0$ such that

$$\sup_{x \in B} |g(x) - T^{\ell, x_0}[g](x)| \leq C_{\ell, \alpha, B} \cdot |g|_{C^{\ell, \alpha}(B)} \quad \text{for all } g \in C^{\ell, \alpha}(B).$$

Proof. Let $\ell \in \mathbb{N}$. Then, by Taylor's theorem for order $\ell - 1$, for all $x \in B$ there exists a $\theta \in (0, 1)$ such that

$$\begin{aligned} g(x) - T^{\ell, x_0}[g](x) &= g(x) - T^{\ell-1, x_0}[g](x) - \sum_{|\nu|=\ell} \frac{\partial^\nu g(x_0)}{\nu!} (x - x_0)^\nu \\ &= \sum_{|\nu|=\ell} \frac{\partial^\nu g(x_0 + \theta(x - x_0))}{\nu!} (x - x_0)^\nu - \sum_{|\nu|=\ell} \frac{\partial^\nu g(x_0)}{\nu!} (x - x_0)^\nu \end{aligned}$$

Now, estimating the right-hand side with the help of $|g|_{C^{\ell, \alpha}(B)}$ results in

$$\begin{aligned} |g(x) - T^{\ell, x_0}[g](x)| &\leq \sum_{|\nu|=\ell} \frac{|\partial^\nu g(x_0 + \theta(x - x_0)) - \partial^\nu g(x_0)|}{|(x_0 + \theta(x - x_0)) - x_0|^\alpha} \frac{\theta^\alpha |x - x_0|^{|\nu|+\alpha}}{\nu!} \\ &\leq C_{\ell, \alpha, B} |g|_{C^{\ell, \alpha}(B)} \end{aligned}$$

for all $x \in B \setminus \{x_0\}$ and $\ell \in \mathbb{N}$. Since this bound obviously holds for $x = x_0$ and for $\ell = 0$ as well, the claim is proven. \blacksquare

Lemma 5.6. Let $B \subset \mathbb{R}^d$ be a non-trivial closed ball and denote its interior by \mathring{B} . Moreover, for $k, \ell \in \mathbb{N}_0$ with $k \leq \ell$, let $\{\mathcal{P}_j^k\}_{j \in \mathbb{N}_0} \subset \Pi_k(B)$ be a sequence of polynomials and suppose that $X(\mathring{B})$ denotes a quasi-Banach space of functions on \mathring{B} , which is continuously embedded into $\mathcal{D}'(\mathring{B})$. Finally, assume that

$$(f_j - \mathcal{P}_j^k) \rightarrow f^1 \quad \text{with respect to } \|\cdot\|_{C^{\ell, \alpha}(B)} \quad \text{and} \quad f_j \rightarrow f \quad \text{in } X(\mathring{B}),$$

as j approaches infinity. Then $f \in C^{\ell, \alpha}(B)$ and

$$f_j \rightarrow f \quad \text{with respect to } \|\cdot\|_{C^{\ell, \alpha}(B)}, \quad \text{as } j \rightarrow \infty.$$

Proof. Since both the spaces $C^{\ell,\alpha}(\mathring{B})$ and $X(\mathring{B})$ are continuously embedded into $\mathcal{D}'(\mathring{B})$, the convergence

$$(f_j - \mathcal{P}_j^k) \rightarrow f^1 \quad \text{and} \quad f_j \rightarrow f$$

takes place in $\mathcal{D}'(\mathring{B})$. Hence, $\mathcal{P}_j^k \rightarrow (f - f^1) \in \mathcal{D}'(\mathring{B})$, as $j \rightarrow \infty$.

On the other hand, the linear space $\Pi_k(\mathring{B})$ of polynomials of degree not larger than k is closed with respect to the convergence (cf. Lemma 5.7 below) in $\mathcal{D}'(\mathring{B})$. Consequently, $f - f^1 =: \mathcal{P}^k \in \Pi_k(B)$ and

$$f = f^1 + \mathcal{P}^k \in C^{\ell,\alpha}(B).$$

Finally, as $|\cdot|_{C^{\ell,\alpha}(B)}$ can not distinguish polynomials of degree less or equal to ℓ ,

$$|f_j - f|_{C^{\ell,\alpha}(B)} = \left| (f_j - \mathcal{P}_j^k) - (f - \mathcal{P}^k) \right|_{C^{\ell,\alpha}(B)} = \left| (f_j - \mathcal{P}_j^k) - f^1 \right|_{C^{\ell,\alpha}(B)} \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

due to our assumption. ■

Lemma 5.7. *Let \mathring{B} denote an open ball in \mathbb{R}^d , $d \in \mathbb{N}$. Then the set of polynomials $\Pi_k(\mathring{B})$ of degree at most $k \in \mathbb{N}_0$ on \mathring{B} is closed with respect to convergence in $\mathcal{D}'(\mathring{B})$.*

Proof. For all $\{\mathcal{P}_j^k\}_{j \in \mathbb{N}_0} \subset \Pi_k(\mathring{B})$ with

$$\mathcal{P}_j^k \rightarrow \mathcal{P} \in \mathcal{D}'(\mathring{B}), \quad \text{as } j \rightarrow \infty,$$

we have to show that $\mathcal{P} \in \Pi_k(\mathring{B})$. We shall prove this statement by induction on $k \in \mathbb{N}_0$. Let $k = 0$. Then $\mathcal{P}_j^0 \equiv a_j \in \mathbb{R}$ is a sequence of constants converging to $\mathcal{P} \in \mathcal{D}'(\mathring{B})$, i.e.,

$$a_j \int_B \varphi(x) \, dx = \int_B \mathcal{P}_j^0(x) \varphi(x) \, dx \rightarrow \mathcal{P}(\varphi) \quad \text{for all } \varphi \in \mathcal{D}(B), \quad \text{as } j \rightarrow \infty.$$

Obviously, the sequence $\{a_j\}_{j \in \mathbb{N}_0}$ has to be bounded in \mathbb{R} and hence there is a subsequence $\{a_{j_\ell}\}_{\ell \in \mathbb{N}_0}$ with $a_{j_\ell} \rightarrow a \in \mathbb{R}$, as $\ell \rightarrow \infty$. By uniqueness of convergence of this subsequence it holds

$$\mathcal{P}(\varphi) = a \int_B \varphi(x) \, dx$$

and thus $\mathcal{P} \equiv a \in \Pi_0(\mathring{B})$.

Let us now assume that $k \in \mathbb{N}$ and that the statement of the lemma is already shown for $0 \leq \ell \leq k - 1$. In addition, let $\nu \in \mathbb{N}_0^d$ with $|\nu| = k$ be a given multi-index. If $\mathcal{P}_j^k \rightarrow \mathcal{P}$ in

$\mathcal{D}'(\mathring{B})$, then also $\partial^\nu \mathcal{P}_j^k \rightarrow \partial^\nu \mathcal{P}$ in $\mathcal{D}'(\mathring{B})$, as $j \rightarrow \infty$. But, for all $j \in \mathbb{N}_0$, $\partial^\nu \mathcal{P}_j^k \equiv a_j^\nu \in \mathbb{R}$ is a polynomial of degree 0. Hence, by the base step of the induction, the sequence $\{\partial^\nu \mathcal{P}_j^k\}_{j \in \mathbb{N}_0}$ converges to some constant a^ν in $\mathcal{D}'(\mathring{B})$. This shows that

$$\mathcal{P}_j^{k-1} := \mathcal{P}_j^k - \sum_{|\nu|=k} \frac{\partial^\nu \mathcal{P}_j^k}{\nu!} x^\nu \quad \text{tends to} \quad \tilde{\mathcal{P}} := \mathcal{P} - \sum_{|\nu|=k} \frac{a^\nu}{\nu!} x^\nu \quad \text{in} \quad \mathcal{D}'(\mathring{B}), \quad \text{as } j \rightarrow \infty.$$

Since \mathcal{P}_j^{k-1} belongs to $\Pi_{k-1}(\mathring{B})$, by induction it follows that $\tilde{\mathcal{P}} \in \Pi_{k-1}(\mathring{B})$, too. Therefore, \mathcal{P} belongs to $\Pi_k(\mathring{B})$ and the proof is complete. ■

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